1 NLPDE I, WS 12/13, Exercise Sheet 3

Due Oct 30 in tutorial session. Note this sheet has two pages (4 exercises)

Exercise 1:

Fix a dyadic grid in $\mathbf{R}^n$, let Q be a dyadic cube in the grid, and let $\mathbf{Q}$ be a collection of dyadic cubes in the grid such that

$$Q \subset \bigcup_{Q' \in \mathbf{Q}} Q'$$

prove (without using measure theory, since this is one of the main ingredients into Lebesgue mesure theory) that

$$|Q| \leq \sum_{Q' \in \mathbf{Q}} |Q'|$$

where $|Q| := l(Q)^n$.

Exercise 2:

Fix a dyadic grid in $\mathbf{R}^n$, let Ω be a bounded open set in $\mathbf{R}^n$, and let $\mathbf{Q}_\Omega$ be the set (Whitney cover) of all maximal dyadic cubes Q of the grid such that $3Q \subset \Omega$.

1. Prove that if $Q, Q' \in \mathbf{Q}_\Omega$ are adjacent, i.e. $\partial Q \cap \partial Q' \neq \emptyset$, then $l(Q) \leq 2l(Q')$.
2. Prove that for each cube $Q \in \mathbf{Q}_\Omega$ we have

$$l(Q) \leq \operatorname{dist}(Q, \Omega^c) \leq 3l(Q)\sqrt{n}$$

Exercise 3:

Let Ω be a bounded open set and $\mathbf{Q}_\Omega$ a Whitney cover as in the above theorem. Let h be a positive harmonic function in Ω.

Prove that the following three are equivalent

1. h is integrable in Ω
2. $\sum_{Q \in \mathbf{Q}_\Omega} |Q| \inf_{x \in Q} h(x)$ is finite
3. $\sum_{Q \in \mathbf{Q}_\Omega} |Q| \sup_{x \in Q} h(x)$ is finite

Exercise 4:

Fix a dyadic grid in $\mathbf{R}^n$. We consider martingales f such that

$$\sup_k \sum_{l(Q)=2^k} |f_Q||Q| < \infty$$

1. Describe one such martingale f such that

$$\sum_{l(Q)=2^k} (\sum_{Q' \subset Q : l(Q')=2^{k-1}} |f_Q - f'_Q||Q'|)$$

   does not converge to 0 as $k \to -\infty$

2. Describe one such martingale that is not of the form

$$f_Q = |Q|^{-1}\nu(Q)$$

   for some finite Borel measure ν on $\mathbf{R}^n$.