1 NLPDE I, WS 12/13, Exercise Sheet 2

Due Oct 25 in tutorial session.

Let Q' be a cube in $\mathbf{R}^{n-1}$ with sidelength $l(Q')$ and let I be an interval with $l(I) \leq l(Q')$. Consider the slab $S = Q' \times I \subset \mathbf{R}^n$ and let f be the indicator function 1_S. Let Γ be the Newton potential in $\mathbf{R}^n$ as in the lecture.

Exercise 1:

Prove or disprove the following estimate: For $x \in 3Q' \times 3I'$ and $x \notin \partial Q' \times I$ we have for some constant C and all $1 \leq i, j \leq n$:

$$|D_i D_j(\Gamma * f)(x)| \leq C(1 + \log |\frac{dist(x, (\partial Q') \times I)}{l(Q')}|)$$

Exercise 2:

Prove or disprove the following estimate:

For $x \notin 3Q' \times 3I'$ we have for some constant C and all $1 \leq i, j \leq n$:

$$|D_i D_j(\Gamma * f)(x)| \leq C\frac{l(I)}{l(Q')}(\frac{dist(x, (\partial Q') \times I)}{l(Q')})^{-n}$$

Exercise 3 :

For a smooth compactly supported function (test function) $\phi : \mathbf{R}^5 \to \mathbf{R}$ define

$$(\Lambda, \phi) := \int_{\mathbf{R}} \int_{\mathbf{R}} x_3 D_1 D_2 \phi(0, 0, x_3, x_4, 0)\, dx_3\ dx_4$$

1. Prove that Λ is a distribution (review the definition of a distribution).
2. Prove that Λ is homogeneous and determine the degree of homogeneity.

Exercise 4:

For every $\alpha \in \mathbf{R}$, construct a non-zero rotation invariant distribution $\Lambda_{(\alpha)}$ on $\mathbf{R}^n$ that is homogeneous of degree α. For which α do we have $(\Lambda_{(\alpha)}, \phi) = 0$ for all ϕ supported outside the ball $B_1(0)$? Prove your assertions.

(Remark: It turns out that $\Lambda_{(\alpha)}$ is unique up to scalar multiple, so that the answer to the second part does not depend on your choice in the first part. You may try to prove uniqueness, though this is not required as part of the exercise.)