1 NLPDE I, WS 12/13, Exercise Sheet 10

Due Jan 22 in tutorial session. This sheet has 3 exercises

Exercise 1:

Review of compact operators:

A bounded operator T from a Hilbert space H to itself is called compact, if for every bounded sequence v_n in H the sequence Tv_n has a convergent subsequence.

1) Prove that finite rank operators are compact.

2) Prove that the set of compact operators is closed under the operator norm, where the operator norm of T is the best possible constant in the a priori estimate

$$\|Tv\| \leq C\|v\|$$

required to hold for every v in the Hilbert space.

3) Prove that Hilbert Schmidt operators are compact. An operator T is Hilbert Schmidt if for some basis e_i of the Hilbert space we have

$$\sum_{i,j} |\langle e_i, He_j\rangle|^2 < \infty$$

where the sum is running over all pairs of basis vectors.

4) Let $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a function in $L^2(\mathbf{R}^{2n})$. Prove that the operator from $L^2(\mathbf{R}^n)$ to itself defined by

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y)f(y)\,dy$$

is compact.

Exercise 2:

Let $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be an integral kernel such that

$$|K(x,y)| \leq \min(|x-y|^{-n+\epsilon}, |x|^{-n-\epsilon}, |y|^{-n-\epsilon})$$

Prove that the operator defined by

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y)f(y)\,dy$$

is a compact operator from $L^2(\mathbf{R}^n)$ to itself.

Let $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ be a smooth compactly supported function. Prove that the operator

$$Kf(x) = \int_{\mathbf{R}^{n-1}} \frac{\phi(x) - \phi(y) - (x-y)\cdot\nabla\phi(y)}{(|x-y|^2 + (\phi(x)-\phi(y))^2)^{n/2}} f(x)\,dx$$

is a compact operator in $L^2(\mathbf{R}^{n-1})$.

Exercise 3:

Let $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ be a smooth compactly supported function. Consider the double layer potential for $r > \phi(P)$

$$\mathcal{K}f(P,r) = \frac{1}{\omega_n} \int_{\mathbf{R}^{n-1}} \frac{r - \phi(Q) - (P-Q) \cdot \nabla\phi(Q)}{(|P-Q|^2 + (\phi(P) - \phi(Q))^2)^{n/2}} f(Q)\, dQ$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. Moreover consider the related singular integral on $\mathbf{R}^{n-1}$

$$Kf(P) = \lim_{\epsilon \to 0} \frac{1}{\omega_n} \int_{|P-Q|>\epsilon} \frac{\phi(P) - \phi(Q) - (P-Q) \cdot \nabla\phi(Q)}{(|P-Q|^2 + (\phi(P) - \phi(Q))^2)^{n/2}} f(Q)\, dQ$$

Prove that for every $P \in \mathbf{R}^{n-1}$

$$\mathcal{K}f(Q) = \frac{1}{2} f(Q) + Kf(Q)$$