1 NLPDE I, WS 12/13, Exercise Sheet 1

Due during week of Oct 15 in tutorial session.

This is a review on harmonic functions, see e.g. Gilbarg Trudinger Chapter 1 or any other source on harmonic functions. Throughout this sheet $n \geq 2$.

Exercise 1 :

Assume f is a continuous function on the sphere $S = \{x : \|x\| = 1\}$ in $\mathbf{R}^n$. Give an explicit integral formula for a function g on the open ball $B = \{x : \|x\| < 1\}$ satisfying

1. g is harmonic in B, i.,e. $\Delta g = 0$ on B.
2. g has a continuous extension to $B \cup S$ which coincides with f on S.

Prove the two properties.

Exercise 2:

Prove that there is at most one function g satisfying the Properties 1) and 2) in Exercise 1.

Exercise 3:

Let Ω be an open subset of $\mathbf{R}^n$ that is symmetric under reflection across the hyperplane $x_n = 0$, that is if $(x_1, \ldots, x_{n-1}, x_n) \in \Omega$ then also $(x_1, \ldots, x_{n-1}, -x_n) \in \Omega$. Assume f is a continuous function on $\{(x_1, \ldots, x_n) \in \Omega : x_n \geq 0\}$ that is harmonic on the set $\{(x_1, \ldots, x_n) \in \Omega : x_n > 0\}$. Prove that this function has a harmonic extension to Ω. (Schwarz reflection principle)

Exercise 4:

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and f a function in $L^1(\Omega)$. Assume that for every smooth function ϕ with compact support inside Ω we have

$$\int_\Omega f(x) \Delta \phi(x)\, dx = 0$$

(That is: f is weakly harmonic) Proof that f is equivalent to a C^∞ function which satisfies

$$\Delta f = 0$$

on Ω (That is: f is harmonic).