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A structure theorem for Harish-Chandra bimodules via coinvariants and Golod rings

Catharina Stroppel[1]

Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW, United Kingdom

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Abstract

We consider the category of Harish-Chandra bimodules for a semisimple complex Lie algebra. We describe algebras of self-extensions of certain simple objects by showing that their blocks are equivalent to module categories over complete intersections or Golod rings. Our main result is a generalisation of Soergel's structural description of the blocks of category $\mathcal{O}$ to a description of the general integral blocks of Harish-Chandra bimodules.

Introduction

The motivation for this paper is the wish to understand the representation theory of complex semisimple Lie groups like $\mathrm{SL}(n, \mathbb{C})$ considered as a reel Lie group. In this context, the category $\mathcal{H}$ of Harish-Chandra bimodules occurs in a natural way and plays a crucial role (see [22,24,25]).

As our main result we prove a generalisation of Soergel's Struktursatz. This implies a ring theoretic description of the category $\mathcal{H}$ providing also a recipe for computing explicitly quivers describing up to Morita-equivalence all the integral blocks of $\mathcal{H}$.

E-mail address: cs@maths.gla.ac.uk.

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To describe extensions between simple objects seems to be a quite difficult problem. Some partial results can be found for example in [4]. Beside our main result we consider the easiest blocks of $\mathcal{H}$ and describe the algebras of (self-)extensions of simple objects via complete intersection and Golod rings.

To make these statements more precise we need some notation. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with universal enveloping algebra $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a fixed Cartan subalgebra. Then $\mathcal{H}$ is the full subcategory of the category of finitely generated $\mathcal{U}$-modules, where the objects are of finite length and locally finite for the adjoint action of $\mathfrak{g}$ (see [22]). The action of the centre $\mathcal{Z}$ of $\mathcal{U}$ gives a decomposition

$$\mathcal{H} = \bigoplus_{\mu,\lambda \in \operatorname{Max} \mathcal{Z}} {}_{\lambda}\mathcal{H}_{\mu}, \tag{0.1}$$

where the summands are indexed by pairs of maximal ideals of $\mathcal{Z}$ (or of central characters) and defined by $\operatorname{Ob}({}_{\lambda}\mathcal{H}_{\mu}) = \{X \in \mathcal{H} \mid \lambda^m X = X\mu^m = 0 \text{ for } m \gg 0\}$.

Using the Harish-Chandra isomorphism in the normalisation of [13, 3.4, 3.5], the maximal ideals of the centre are in bijection with the dominant weights $\lambda \in \mathfrak{h}^\star$, hence we can also index the blocks ${}_{\lambda}\mathcal{H}_{\mu}$ by dominant weights λ and μ. (Note that we call a weight $\lambda \in \mathfrak{h}^*$ *dominant* if $\langle \lambda + \rho, \check{\alpha} \rangle \geqslant 0$ for any simple coroot $\check{\alpha}$, where ρ denotes the half-sum of positive roots.)

The simple objects of these categories are classified ([26], see also [5, Theorem 5.6], [13, 6.29]). Unfortunately, these categories do not have enough projectives. Therefore it makes sense to study (for any fixed positive integer n) a 'truncated' version ${}_{\lambda}\mathcal{H}_{\mu}^{n}$ of ${}_{\lambda}\mathcal{H}_{\mu}$. A very natural truncation is given by the following set of objects

$$\operatorname{Ob}\bigl({}_{\lambda}\mathcal{H}_{\mu}^{n}\bigr) = \{X \in {}_{\lambda}\mathcal{H}_{\mu} \mid X\mu^n = 0\}.$$

This ensures enough projectives and ${}_{\lambda}\mathcal{H}_{\mu}$ is the limit of all this full subcategories. In particular, the structure of ${}_{\lambda}\mathcal{H}_{\mu}$ is determined by homomorphisms between projective objects in ${}_{\lambda}\mathcal{H}_{\mu}^{n}$.

For $\lambda, \mu \in \mathfrak{h}^*$ dominant and integral we consider an exact functor $\mathbb{V} : {}_{\lambda}\mathcal{H}_{\mu}^{n} \to \mathcal{Z} \otimes \mathcal{Z}$-mof which annihilates all simple modules except the one with maximal Gelfand–Kirillov dimension in its block. This generalises Soergel's combinatorial functor $\mathcal{O}_0 \to \text{mof-}C$. As a corollary of our Theorem 4.1 we get the following structure theorem generalising [22, Theorem 13].

Theorem 1. *Let $\lambda, \mu \in \mathfrak{h}^*$ be dominant and integral. Let $P, Q \in {}_{\lambda}\mathcal{H}_{\mu}^{n}$ be projective. Then $\mathbb{V}$ induces an isomorphism*

$$\operatorname{Hom}_{\mathcal{H}}(Q, P) \cong \operatorname{Hom}_{\mathcal{Z}\otimes\mathcal{Z}}(\mathbb{V}Q, \mathbb{V}P).$$

In particular, this gives a combinatorial description of the categories of Harish-Chandra bimodules with generalised integral central character from both sides.

In the case $n = 1$, Bernstein and Gelfand [5, Theorem 5.9] proved an equivalence of categories

$$\mathcal{T}_{(\lambda,\mu)} : {}_\lambda\mathcal{H}^1_\mu \xrightarrow{\sim} \mathcal{O}_\lambda \tag{0.2}$$

when μ is regular and dominant. (Here $\mathcal{O}$ denotes the well-known category defined by Bernstein, Gelfand and Gelfand in [6] for a semisimple Lie algebra $\mathfrak{g}$ with a fixed Borel $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$. Its objects are finitely generated $\mathcal{U}(\mathfrak{g})$-modules of finite length, on which $\mathfrak{h}$ acts diagonally and $\mathcal{U}(\mathfrak{b})$ locally nilpotent.) For μ singular, the functor $\mathcal{T}_{(\lambda,\mu)}$ is still faithful; its image is described in [5, 5.9], [13, 6.18].

The equivalence (0.2) implies that the extensions between simple objects of ${}_\lambda\mathcal{H}^1_\mu$ are given by Kazhdan–Lusztig theory, more precisely

$$\dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathcal{O}}\big(L(x\cdot\lambda), L(y\cdot\lambda)\big) = \sum_{w\in W,\, j\in\mathbb{Z}} \alpha_{(x,w,j)}\alpha_{(y,w,i-j)}, \tag{0.3}$$

where $p_{x,y} = \sum_j \alpha(x, y, j)t^j$ is a certain Kazhdan–Lusztig polynomial (see, e.g., [8, Theorems 1.1.3 and 2.12.6] or [23] for an explicit formula). This indicates that it should be almost impossible to find a general formula for $\dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathcal{O}}(L, L')$, if L and L' are simple objects in ${}_\lambda\mathcal{H}^n_\mu$ for arbitrary $n \geqslant 1$. Nevertheless we will give some answers for very special cases, i.e. when $(\lambda, \mu) \in \{(-\rho, -\rho), (-\rho, 0), (0, -\rho)\}$ and n is arbitrary by describing ${}_\lambda\mathcal{H}^n_\mu$ via a module category over a (generalised) ring of coinvariants. Our results, although not as general as [10], are rather explicit.

Let us for the moment consider ${}_0\mathcal{H}^1_{-\rho}$ having only one simple object ${}_0L_{-\rho}$. The Bernstein–Gelfand functor from (0.2) and Soergel's Endomorphism Theorem ([21, 2.2]) show that ${}_0\mathcal{H}^1_{-\rho} \xrightarrow{\sim}$ mof-C, where mof-C denotes the category of finitely generated modules over the coinvariant algebra $C = S(\mathfrak{h})/((S(\mathfrak{h})^+)^W)$ defined by the Weyl group W of $\mathfrak{g}$, and where $S(\mathfrak{h})$ denotes the algebra of regular functions on $\mathfrak{h}^*$. The first result in the paper follows then from the theory of complete intersection rings: There is an isomorphism

$$\operatorname{Ext}^\bullet_{{}_0\mathcal{H}^1_{-\rho}}({}_0L_{-\rho}, {}_0L_{-\rho}) \cong \operatorname{Ext}^\bullet_C(\mathbb{C}, \mathbb{C}) \cong S(\mathfrak{h})$$

as graded vector spaces (Theorem 1.3). The algebra structure can be described explicitly by a theorem of Sjoedin (Theorem 1.4). These results have also an application to the representation theory in positive characteristic (Theorem 1.6).

If we consider ${}_0\mathcal{H}^n_{-\rho}$ for arbitrary $n > 1$, then this category is no longer a category of modules over a complete intersection ring (at least for $\mathfrak{g} \neq \mathfrak{sl}_2$), but nevertheless a category of modules over a 'generalised' algebra of coinvariants (Theorem 2.2):

$${}_0\mathcal{H}^n_{-\rho} \cong \text{mof-}S \otimes_{S^W} S^W\big/\tilde{\mathfrak{m}}^n,$$

where $\tilde{\mathfrak{m}}$ is a certain maximal ideal of S^W. A description of the algebra of self-extensions of the simple object in ${}_0\mathcal{H}^n_{-\rho}$ follows then from the theory of Golod rings (see, e.g., [2]). In particular, the categories become accessible via computer algebra software.

The paper is organised as follows: The main part of the paper starts in Section 2 with the definition of the structure functor $\mathbb{V}$, its behaviour with translation functors and its faithfulness on projectives. The full structure theorem will be proved in Section 4. Sections 1 and 2 contain explicit results in the case where $\mathbb{V}$ defines an equivalence of categories to module categories over coinvariants and Golod rings. The remaining Section 3 contains a warning, since we show that the BG-equivalence does not generalise to 'generalised singular' blocks in the way one might expect.

1. Harish-Chandra bimodules

Let $\mathfrak{g}$ be a semisimple complex Lie algebra with a fixed Borel subalgebra $\mathfrak{b}$ and a fixed Cartan subalgebra $\mathfrak{h}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding triangular decomposition. Let $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{b})$ and $S = \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h})$ be the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{h}$, respectively. Let $\mathcal{Z} \subset \mathcal{U}$ be the centre. Let ρ be the half-sum of positive roots. We denote by W the Weyl group and for any $\lambda \in \mathfrak{h}^*$ let $W_\lambda = \{w \in W \mid w(\lambda + \rho) - \rho) = \lambda\}$ be the stabiliser for the 'dot-action' defined as $w \cdot \lambda = w(\lambda + \rho) - \rho$.

We consider the category $\mathcal{H}$ of Harish-Chandra bimodules. The objects are finitely generated $\mathcal{U}$-bimodules of finite length which are locally finite for the adjoint action of $\mathfrak{g}$ (see, e.g., [13,22]). The morphisms are the morphisms of $\mathcal{U}$-bimodules. Let $\operatorname{Max} \mathcal{Z}$ be the set of maximal ideals in $\mathcal{Z}$. The action of $\mathcal{Z}$ gives the decomposition (0.1) from the introduction. The categories ${}_\lambda\mathcal{H}_\mu^n$ have enough projectives. More generally, let $I \lhd \mathcal{Z}$ be an ideal of finite codimension and denote by $\mathcal{H}^I$ the full subcategory of $\mathcal{H}$ given by all objects which are annihilated by I from the right-hand side. The subcategory ${}_\mu\mathcal{H}^I$ is given by objects with generalised central character $\ker \chi_\mu$ from the left-hand side.

Theorem 1.1. *The projective objects in $\mathcal{H}^I$ are the direct summands of modules of the form $E \otimes \mathcal{U}/(\mathcal{U}I)$, where E is a finite-dimensional $\mathfrak{g}$-module with trivial right $\mathfrak{g}$-action.*

Proof. Mutatis mutandis [13, 6.14]. □

Let $\xi : \mathcal{Z} \to S$ be the Harish-Chandra homomorphism, normalised such that it induces a surjective map $\mathfrak{h}^* \to \operatorname{Max} \mathcal{Z} : \lambda \mapsto \chi_\lambda$, which is constant on orbits of the Weyl group action with fix point $-\rho$. In this note, we first consider blocks ${}_\lambda\mathcal{H}_\mu$ for $(\lambda, \mu) \in \{(-\rho, -\rho), (0, -\rho), (-\rho, 0)\}$. Each of these blocks has only one simple object [13, 6.23, 6.26]; we denote it by ${}_\lambda L_\mu \in {}_\lambda\mathcal{H}_\mu$. We want to describe $\operatorname{Ext}^\bullet_{{}_\lambda\mathcal{H}_\mu^n}({}_\lambda L_\mu, {}_\lambda L_\mu)$ for such blocks and arbitrary n. For any ring R we denote by R-mof (or mof-R) the category of finitely generated left (or right) R-modules.

1.1. Harish-Chandra bimodules and coinvariants

Let us first consider the category ${}_0\mathcal{H}_{-\rho}^1$. Via the equivalence (0.2), the only indecomposable projective object in ${}_0\mathcal{H}_{-\rho}^1$ is mapped to the projective cover $P_{w_o}^1$ of the simple Verma module in $\mathcal{O}_0$. Hence, by [21, Endomorphismensatz], we get equivalences of categories

$$ {}_0\mathcal{H}^1_{-\rho} \xrightarrow{\sim} \text{mof-End}\big(P^1_{w_o}\big) \xrightarrow{\sim} \text{mof-}C, \tag{1.1}$$

where $C = S(\mathfrak{h})/(S(\mathfrak{h})^W_+)$ is the coinvariant algebra having the following nice properties:

Lemma 1.2. *Let C be the algebra of coinvariants for any semisimple complex Lie algebra $\mathfrak{g}$ (or more general C is an algebra of coinvariants for a finite pseudo reflection group acting linearly on a finite dimensional complex vector space). It has the following properties*:

(1) Krull-dim $C = 0$.
(2) *C is a complete intersection ring.*
(3) *C is Gorenstein (i.e.* $\operatorname{Ext}^i_C(\mathbb{C}, C) \cong \mathbb{C}$ *for $i = 0$ and $= 0$ for $i \neq 0$.)*
(4) *C is Cohen–Macaulay (i.e.* depth $C =$ Krull-dim C).

Proof. Let $\mathfrak{m}$ be the maximal ideal of C and let $\mathfrak{p} \subseteq \mathfrak{m}$ be a prime ideal. Since C is a positively graded ring of finite dimension (see [15, 23.1] or [7, V, 5.2, Théorème 1]), for each $x \in \mathfrak{m}$ there is a $n \in \mathbb{N}$ such that $x^n = 0 \in \mathfrak{p}$. Hence $x \in \mathfrak{p}$, which implies $\mathfrak{p} = \mathfrak{m}$. Therefore, the Krull dimension of C is zero. A noetherian local ring $(R, \mathfrak{m}, \mathbb{K})$ is a *complete intersection ring*, say $R \cong S/I$ for some regular ring S, if and only if Krull-dim $R = \nu(\mathfrak{m}) - \nu(I)$, where $\nu(\bullet)$ denotes the cardinality of a minimal system of generators (see [9, Theorem 2.3.3]). Hence, it is enough to show that $\dim_{\mathbb{C}} \mathfrak{h}$ is equal to the minimal number of generators of $((S(\mathfrak{h})^+)^W)$. This is [7, V, 5.2, Théorème 3]. For the remaining statements (1.2) and (1.2), see [9, Proposition 3.1.20]. □

We get our first result.

Theorem 1.3. *There is an isomorphism of graded vector spaces*

$$\operatorname{Ext}^\bullet_{{}_0\mathcal{H}^1_{-\rho}}({}_0L_{-\rho}, {}_0L_{-\rho}) \cong S(\mathfrak{h}),$$

where $S(\mathfrak{h})$ has the usual grading, such that $S(\mathfrak{h})^1 = \mathfrak{h}$.

Proof. Set $E := \operatorname{Ext}^\bullet_{{}_0\mathcal{H}^1_{-\rho}}({}_0L_{-\rho}, {}_0L_{-\rho})$. By equivalence (1.1), $E \cong \operatorname{Ext}^\bullet_C(\mathbb{C}, \mathbb{C})$. Let $P(t) = P^C_{\mathbb{C}}(t) := \sum_{i=0}^\infty \dim_{\mathbb{C}} \operatorname{Ext}^n_C(\mathbb{C}, \mathbb{C}) t^n$ be the corresponding Poincaré series. The deviations $\epsilon_n \in \mathbb{Z}$ are uniquely defined by the equality

$$P(t) = \prod_{i=1}^\infty \big(1 + t^{2i-1}\big)^{\epsilon_{2i-1}} \Big/ \prod_{i=1}^\infty \big(1 - t^{2i}\big)^{\epsilon_{2i}}$$

of power series. Since, by the previous lemma, C is a complete intersection ring, $\epsilon_n(R) = 0$ for $n \geqslant 3$ (see [1, Theorem 7.3.3]). Moreover, $\epsilon_1 = \epsilon_2 = \dim_{\mathbb{C}} \mathfrak{h}$ ([1, 7.1.5] and [7, V, 5.2, Théorème 3]). Therefore, the Betti numbers for the trivial module are given by the following formula:

$$P(t)=\frac{(1+t)^{\epsilon_1}}{(1-t^2)^{\epsilon_2}}=\frac{(1+t)^{\epsilon_1-\epsilon_2}}{(1-t)^{\epsilon_2}}=\underbrace{\left(1+t+t^2+\cdots\right)^{\epsilon_2}}_{A}(1+t)^{\epsilon_1-\epsilon_2}.$$

The coefficient of t^n in A is the number of sequences $(\alpha_1,\alpha_2,\dots,\alpha_{\epsilon_2})\in\mathbb{Z}_{\geqslant 0}^{\epsilon_2}$ such that $\sum_{i=1}^{\epsilon_2}\alpha_i=n$. On the other hand, these sequences index a basis of $S(\mathfrak{h})^n$ given by the polynomials $\prod_{i=1}^{\epsilon_2}h_i^{\alpha_i}$, where $\{h_i\}_{1\leqslant i\leqslant\epsilon_2}$ is a basis of $\mathfrak{h}$. In particular, there is an isomorphism of graded vector spaces $\mathrm{Ext}_C^\bullet(\mathbb{C},\mathbb{C})\cong S(\mathfrak{h})$. □

The algebra structure is given by the following theorem.

Theorem 1.4 [18]. *Let $C\cong\mathbb{C}[x_1,\dots,x_n]/(f_1,\dots,f_n)$ be a ring of coinvariants, with a minimal set $f_1,\dots,f_n$ of generators for the ideal generated by the invariants without a constant term. Let $a_{hi,j}\in\mathbb{C}[x_1,\dots,x_n]$ (for $1\leqslant i,j\leqslant n$) be defined by*

$$f_j=\sum_{1\leqslant h\leqslant i\leqslant n}a_{hi,j}x_hx_i.$$

Then there is an isomorphism of graded algebras

$$\mathrm{Ext}_C^\bullet(\mathbb{C},\mathbb{C})\cong\mathcal{U}_{\mathbb{Z}}(\mathfrak{p})$$

for some graded Lie algebra $\mathfrak{p}$ such that the following holds:

(1) $\dim_{\mathbb{C}}\mathfrak{p}^{(i)}=n$ *for $i\in\{0,1\}$ and 0 otherwise. In particular,* $\mathrm{Ext}_C^\bullet(\mathbb{C},\mathbb{C})$ *is generated by its elements of degree at most* 2.
(2) *There is an ordered basis $\{\theta_i\}_{1\leqslant i\leqslant n}$ of $\mathfrak{p}^{(1)}$ such that*

$$[\theta_h,\theta_i]=-\sum_{j=1}^{r}\pi(a_{hi,j})\theta_{n+j}\quad\textit{for } h<i\quad\textit{and}$$

$$[\theta_i,\theta_i]=-2\sum_{j=1}^{r}\pi(a_{ii,j})\theta_{n+j}\quad\textit{for all } i,$$

where $\pi:\mathbb{C}[x_1,\dots,x_n]\twoheadrightarrow\mathbb{C}$ denotes the evaluation morphism at 0.

Corollary 1.5. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $n>1$ with corresponding algebra of coinvariants C. Then the algebra* $\mathrm{Ext}_C^\bullet(\mathbb{C},\mathbb{C})$ *is not commutative.*

Proof. By the formulae above it is sufficient to show that in some minimal set of generators of the ideal generated by invariant polynomials without constant term, there is a homogeneous element of degree two. If we assume the contrary then $\dim_{\mathbb{C}}C^{(2)}=\binom{n+1}{n-1}$ is greater then the number of elements of length two in the Weyl group (which is always $n(n+1)/2-1$). This is a contradiction, since $\dim_{\mathbb{C}}C^j$ is given by the number of elements in the Weyl group of length j. (Note that C is the cohomology ring of the corresponding flag variety.) $\square$

Theorem 1.4 can also be applied to the representation theory of Lie algebras over fields with positive characteristic: Let k be an algebraically closed field of characteristic $p > 0$. Let G be a reductive algebraic group defined over k with simply connected commutator subgroup. Let $(\mathfrak{g}, \mathfrak{h})$ denote its Lie algebra with universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We assume that the Killing form is not degenerated on $[\mathfrak{g}, \mathfrak{g}]$. Let $\chi \in \mathfrak{g}^*$ be regular nilpotent and of Standard Levi form (see [11, Definition 3.1]). Let $\mathcal{U}(\mathfrak{g})_\chi$ denote the corresponding restricted universal enveloping algebra, i.e. $\mathcal{U}_\chi(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/\langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\rangle$, where $x^{[p]}$ denotes the pth power in $\mathfrak{g}$. Let $\mathcal{C}$ denote the category of $\mathcal{U}_\chi(\mathfrak{g})$ modules as defined in [3]. Then the following holds:

Theorem 1.6. *Let $\lambda \in \mathfrak{h}^*$ such that $\lambda(h) - \lambda(h^{[p]}) = \chi(h)^p$ holds for all $h \in \mathfrak{h}$. Assume λ to be regular. Let $L_\chi(\lambda) = \mathcal{U}(\mathfrak{g})_\chi \otimes_{\mathcal{U}_\chi(\mathfrak{b})} k_\lambda$ denote the simple Baby–Verma module with highest weight λ. Then*

$$\mathrm{Ext}^\bullet_{\mathcal{C}}\big(L_\chi(\lambda), L_\chi(\lambda)\big) \cong S(\mathfrak{h})$$

as graded vector spaces. The multiplication is given by Theorem 1.4.

Proof. Since $L_\chi(\lambda)$ is the unique simple object in its block (see [11, Theorem 2.4]), its projective cover $Q_\chi(\lambda)$ is a projective generator of its block. According to [3, 19.8] (or [14, 10.12]) there is a natural isomorphism $\mathrm{End}_{\mathfrak{g}}(Q_\chi(\lambda)) \cong S(\mathfrak{h})/((S(\mathfrak{h})^+)^W)$. Therefore the theorem follows by Morita equivalence. $\square$

Remark 1.7. The theorem of Friedlander and Parshall [11, Theorem 2.4] also says that the projective module $Q_\chi(\lambda)$ has length $|W \cdot \lambda|$, where W is the Weyl group of G. This module gives therefore an example of a self-extension of $L_\chi(\lambda)$ of length $|W \cdot \lambda|$.

2. Module categories over Golod rings

Before considering ${}_0\mathcal{H}^n_{-\rho}$ for arbitrary n, let us first look at the 'most singular' case

Theorem 2.1. (1) *There is an isomorphism of algebras*

$$E := \mathrm{Ext}^\bullet_{{}_{-\rho}\mathcal{H}^n_{-\rho}}({}_{-\rho}L_{-\rho}, {}_{-\rho}L_{-\rho}) \cong \mathrm{Ext}_{S/\mathfrak{m}^n}(\mathbb{C}, \mathbb{C})$$

for some maximal ideal $\mathfrak{m} \subset S = S(\mathfrak{h})$.

(2) *As an algebra, E is finitely generated and finitely presentable. It is generated by E^j for $j \leqslant 2$.*

(3) *If $\dim_{\mathbb{C}} \mathfrak{h} = 1$ and $n > 1$ then $\dim_{\mathbb{C}} E^i = 1$ for all $i \in \mathbb{N}$.*

(4) *If $\dim_{\mathbb{C}} \mathfrak{h} > 1$ and $n > 1$ then $\{\dim_{\mathbb{C}} E^i\}_{i \geqslant 0}$ is of exponential growth.*

Proof. (1) By Theorem 1.1, $\mathcal{U}/\mathcal{U}(\ker\chi_{-\rho}^n)$ is a (the only) indecomposable projective object in ${}_{-\rho}\mathcal{H}_{-\rho}^n$. Hence, ${}_{-\rho}\mathcal{H}_{-\rho}^n \cong \text{mof-End}_{\mathcal{H}}(\mathcal{U}/\mathcal{U}(\ker\chi_{-\rho}^n))$. On the other hand, there is an obvious isomorphism of rings

$$\begin{aligned} \operatorname{End}_{\mathcal{H}}\bigl(\mathcal{U}/\mathcal{U}(\ker\chi_{-\rho})^n\bigr) &\cong \mathcal{Z}/(\ker\chi_{-\rho})^n \\ \phi &\mapsto \phi(1). \end{aligned} \tag{2.1}$$

Since the centre is a polynomial ring in $\dim_{\mathbb{C}}\mathfrak{h}$ variables, the first part of the theorem follows.

(2) Since $S/\mathfrak{m}^n$ is a Golod ring (see [1, Theorem 5.2.4], [17] or [19]). The assertions are given by [19, Theorems 2 and 3].

(3) In the case where $\mathfrak{g} = \mathfrak{sl}_2$, $S/\mathfrak{m}^n$ is a hypersurface (i.e. codepth $S/\mathfrak{m}^n \leqslant 1$), we can compute an explicit periodic resolution:

$$\cdots \xrightarrow{1\mapsto\bar{x}} \mathbb{C}[x]/(x)^n \xrightarrow{1\mapsto\bar{x}^{n-1}} \mathbb{C}[x]/(x)^n \xrightarrow{1\mapsto\bar{x}} \mathbb{C}[x]/(x)^n \twoheadrightarrow \mathbb{C}.$$

(4) See [1, (5.0.1)]. □

Concrete formulae can be found in [2, Lemma 6.6].

Let $\lambda, \mu \in \mathfrak{h}^*$ be dominant and integral. We denote by ${}_\lambda L_\lambda \in {}_\mu\mathcal{H}_\lambda$ the simple object with maximal Gelfand–Kirillov dimension. Let ${}_\mu P_\lambda^n \in {}_\mu\mathcal{H}_\lambda^n$ be its projective cover and, more general, let ${}_\mu P^I$ be its projective cover in ${}_\mu\mathcal{H}^I$ for any proper $\ker\chi_\lambda$-primary ideal $I \lhd \mathcal{Z}$ of finite codimension. We denote such an ideal I by $I \overset{\lambda}{\lhd} \mathcal{Z}$. Let $\widehat{S} = \widehat{S_{(\mathfrak{h})}}$ be the completion of S at the ideal generated by $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$ we denote by $\lambda^\#$ the endomorphism of S induced via translation by λ. Given a maximal ideal $\Lambda = \ker\chi_\lambda$ of the centre $\mathcal{Z} \subset \mathcal{U}$, the completion at this ideal defines an injective homomorphism

$$\lambda^\# \circ \xi : \widehat{\mathcal{Z}_\Lambda} \to \widehat{S} \tag{2.2}$$

for each $\lambda \in \mathfrak{h}^*$, which is even an isomorphism if λ is regular (since in this case S^W maps surjectively onto S/Λ^n). In any case the image is $\widehat{S}^{W_\lambda}$ with maximal ideal $\mathfrak{m}_\lambda$. Let $\hat{I}$ denote the ideal induced by $I \overset{\lambda}{\lhd} \mathcal{Z}$. We abbreviate $\widehat{S}^\lambda = \widehat{S}^{W_\lambda}$. The following theorem describes the blocks we are mainly interested in as module categories over Golod rings.

Theorem 2.2. *Let $(\lambda, \mu) \in \{(0, -\rho), (-\rho, 0), (-\rho, -\rho)\}$. There is an equivalence of categories*

$${}_\mu\mathcal{H}_\lambda^n \cong \widehat{S}^\mu \otimes_{\widehat{S}^W} \bigl(\widehat{S}^\lambda/\mathfrak{m}_\lambda^n\bigr)\text{-mof}\,.$$

To prove the theorem we need some preparation. Let $\mathbb{V}_{(\lambda,\mu)}$ be the exact functor

$$\mathbb{V}_{(\lambda,\mu)} : {}_\lambda\mathcal{H}_\mu \to \mathbb{C}\text{-mof}$$

which is defined (up to equivalence) by the properties $\dim_{\mathbb{C}}({}_\lambda L_\mu) = 1$ and $\dim_{\mathbb{C}} L' = 0$ for all other simple objects L': We choose projective covers ${}_\lambda P^n_\mu \in {}_\lambda\mathcal{H}^n_\mu$ of ${}_\lambda L_\mu$ and projections $p_{n,m} : {}_\lambda P^n_\mu \twoheadrightarrow {}_\lambda P^m_\mu$ for $n \geqslant m$ such that $({}_\lambda P^n_\mu, p_{n,m})$ becomes a projective system. The functor is then given as

$$\mathbb{V}_{(\lambda,\mu)} X := \varinjlim \operatorname{Hom}_{\mathcal{H}}\big({}_\lambda P^n_\mu, X\big)$$

for $X \in {}_\lambda\mathcal{H}_\mu$. The action of the centre on X defines a $\mathcal{Z} \otimes \mathcal{Z}$-bimodule structure on $\mathbb{V}_{(\lambda,\mu)} X$. Hence, by completion, we have a functor

$$\mathbb{V}_{(\lambda,\mu)} : {}_\lambda\mathcal{H}_\mu \to \widehat{S}^\lambda \otimes \widehat{S}^\mu\text{-mof}$$

(For $\lambda = \mu = 0$, this functor is defined in [22].) Let λ', μ' be dominant and integral weights and let $\mathrm{pr}_{(\mu,\mu')}$ denote the projection onto ${}_\mu\mathcal{H}_{\mu'}$. Then the *translation functor* $\theta^{(\mu,\mu')}_{(\lambda,\lambda')}$ is defined as follows:

$$\begin{aligned} \theta^{(\mu,\mu')}_{(\lambda,\lambda')} : {}_\lambda\mathcal{H}_{\lambda'} &\to {}_\mu\mathcal{H}_{\mu'} \\ X &\mapsto \mathrm{pr}_{(\mu,\mu')}\big(X \otimes E(\mu-\lambda)^l \otimes E(\mu'-\lambda')^r\big), \end{aligned}$$

where $E(\mu-\lambda)$ stands for the finite-dimensional, irreducible $\mathfrak{g}$-module with extremal weight $(\mu-\lambda)$. The upper index l (or r) indicates that $E(\mu-\lambda)$ is considered as a left (or right) $\mathfrak{g}$-module and becomes a $\mathcal{U}(\mathfrak{g})$-bimodule with trivial right (or left) action. Let us denote by θ^l_s and θ^r_s the translation functors through the s-wall; more precisely: we choose λ' and $\mu' \in \mathfrak{h}^*$ such that $\lambda - \lambda'$ and $\mu - \mu'$ are integral and $W_{\lambda'} = W_{\mu'} = \{1, s\}$. We set

$$\theta^l_s := \theta^{(\lambda,\mu)}_{(\lambda',\mu)} \circ \theta^{(\lambda',\mu)}_{(\lambda,\mu)} : {}_\lambda\mathcal{H}_\mu \to {}_\lambda\mathcal{H}_\mu \quad \text{and} \quad \theta^r_s := \theta^{(\lambda,\mu)}_{(\lambda,\mu')} \circ \theta^{(\lambda,\mu')}_{(\lambda,\mu)} : {}_\lambda\mathcal{H}_\mu \to {}_\lambda\mathcal{H}_\mu.$$

(Up to equivalence, these functors do not depend on the choice of λ' and μ'. For details, see [12,13].) Translation through the wall is 'compatible' with $\mathbb{V}$ in the following way.

Lemma 2.3. (1) *Let λ, μ, ν be dominant and integral weights and let us assume $W_\mu \supseteq W_\lambda$; if $I \overset{\nu}{\lhd} \mathcal{Z}$ then*

$$\theta^{(\lambda,\nu)}_{(\mu,\nu)}\big({}_\mu P^I\big) \cong {}_\lambda P^I.$$

(2) *Let λ, μ, ν be dominant and integral weights, let $W_\lambda \subseteq W_\mu$. There is a natural equivalence of functors* (*with* res *the functor restricting the scalars*):

$$\mathbb{V}\theta^{(\nu,\lambda)}_{(\nu,\mu)}(\bullet) \cong \big(\widehat{S}^\nu \otimes \widehat{S}^\lambda\big) \otimes_{\widehat{S}^\nu \otimes \widehat{S}^\mu} \mathbb{V}(\bullet), \tag{2.3}$$

$$\mathbb{V}\theta^{(\lambda,\nu)}_{(\mu,\nu)}(\bullet) \cong \big(\widehat{S}^\lambda \otimes \widehat{S}^\nu\big) \otimes_{\widehat{S}^\mu \otimes \widehat{S}^\nu} \mathbb{V}(\bullet). \tag{2.4}$$

$$\mathbb{V}\theta^{(\nu,\mu)}_{(\nu,\lambda)}(\bullet) \cong \mathrm{res}^{(\widehat{S}^\nu \otimes \widehat{S}^\lambda)}_{(\widehat{S}^\nu \otimes \widehat{S}^\mu)} \mathbb{V}(\bullet), \tag{2.5}$$

$$\mathbb{V}\theta^{(\mu,\nu)}_{(\lambda,\nu)}(\bullet) \cong \mathrm{res}^{(\widehat{S}^\lambda \otimes \widehat{S}^\nu)}_{(\widehat{S}^\mu \otimes \widehat{S}^\nu)} \mathbb{V}(\bullet). \tag{2.6}$$

Proof. (1) The bimodule on the left-hand side is obviously projective in ${}_\lambda\mathcal{H}^I$ by adjointness properties of translation functors. Let L' be a simple object in ${}_\lambda\mathcal{H}^I$. The adjointness properties of translation functors give

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{H}}\big(\theta_{(\mu,\nu)}^{(\lambda,\nu)}{}_\mu P^I, L'\big) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^I, \theta_{(\lambda,\nu)}^{(\mu,\nu)} L'\big) = 0$$

unless L' has maximal Gelfand–Kirillov dimension (see [13, 4.12(3), 9.1(3)]), in which case $\theta_{(\lambda,\nu)}^{(\mu,\nu)} L' \cong {}_\mu L_\nu$; hence the space in question is one-dimensional.

(2) The formulae (2.4) and (2.6) are just reformulations of [22, Theorem 12 and Proposition 6] in the case ν is regular.

With the assumptions of the lemma, let $I \overset{\nu}{\triangleleft} \mathcal{Z}$. Let $X \in {}_\lambda\mathcal{H}^I$. We have natural isomorphisms

$$\mathbb{V}\theta_{(\lambda,\nu)}^{(\mu,\nu)} X \cong \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^I, \theta_{(\lambda,\nu)}^{(\mu,\nu)} X\big) \cong \operatorname{Hom}_{\mathcal{H}}\big(\theta_{(\mu,\nu)}^{(\lambda,\nu)}{}_\mu P^I, X\big) \cong \operatorname{Hom}_{\mathcal{H}}\big({}_\lambda P^I, X\big) = \mathbb{V}X$$

of vector spaces. Let now $\nu' \in \mathfrak{h}^*$ be a dominant, integral and regular weight. Let $J \overset{\nu'}{\triangleleft} \mathcal{Z}$ be the annihilator of $\theta_{(\mu,\nu)}^{(\mu,\nu')}{}_\mu P^I$ as right $\mathcal{Z}$-module. Since $\operatorname{Hom}_{\mathcal{H}}(\theta_{(\mu,\nu)}^{(\mu,\nu')}{}_\mu P^I, L) \cong \operatorname{Hom}_{\mathcal{H}}({}_\mu P^I, \theta_{(\mu,\nu')}^{(\mu,\nu)} L) \cong \mathbb{C}$ for $L = {}_\mu L_{\nu'}$ and zero for any simple object $L \in {}_\mu\mathcal{H}_{\nu'}$ having non-maximal Gelfand–Kirillov dimension, there is a surjection ${}_\mu P^J \twoheadrightarrow \theta_{(\mu,\nu)}^{(\mu,\nu')}{}_\mu P^I$. It induces an inclusion

$$\operatorname{Hom}_{\mathcal{H}}\big(\theta_{(\mu,\nu)}^{(\mu,\nu')}{}_\mu P^I, \theta_{(\mu,\nu)}^{(\mu,\nu')} X\big) \hookrightarrow \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^J, \theta_{(\mu,\nu)}^{(\mu,\nu')} X\big). \tag{2.7}$$

Let M be the minimal bisubmodule of X such that X/M has not maximal Gelfand–Kirillov dimension. Our definitions imply that the projective cover of $\theta_{(\mu,\nu)}^{(\mu,\nu')} M \in {}_\mu\mathcal{H}^J$ is a direct sum of copies of ${}_\mu P^J$. We get

$$\begin{aligned}
\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^J, \theta_{(\mu,\nu)}^{(\mu,\nu')} X\big) &= \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^J, \theta_{(\mu,\nu)}^{(\mu,\nu')} M\big) = \big[\theta_{(\mu,\nu)}^{(\mu,\nu')} M : {}_\mu L_{\nu'}\big] \\
&= \big[\theta_{(\mu,\nu)}^{(\mu,\nu')} X : {}_\mu L_{\nu'}\big] = |W_\nu|[X : {}_\mu L_\nu] \\
&= \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^I, \theta_{(\mu,\nu')}^{(\mu,\nu)}\theta_{(\mu,\nu)}^{(\mu,\nu')} X\big) \\
&= \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{H}}\big(\theta_{(\mu,\nu)}^{(\mu,\nu')}{}_\mu P^I, \theta_{(\mu,\nu)}^{(\mu,\nu')} X\big).
\end{aligned}$$

This shows that (2.7) is in fact an isomorphism. We get the following natural isomorphisms of left $\widehat{S}^\mu$-modules:

$$\begin{aligned}
\mathbb{V}\theta_{(\lambda,\nu)}^{(\mu,\nu)} \bigoplus_{i=1}^{|W_\nu|} X &\overset{(1)}{\cong} \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^I, \theta_{(\lambda,\nu)}^{(\mu,\nu)}\theta_{(\lambda,\nu')}^{(\lambda,\nu)}\theta_{(\lambda,\nu)}^{(\lambda,\nu')} X\big) \overset{(2)}{\cong} \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^I, \theta_{(\mu,\nu')}^{(\mu,\nu)}\theta_{(\lambda,\nu')}^{(\mu,\nu')}\theta_{(\lambda,\nu)}^{(\lambda,\nu')} X\big) \\
&\overset{(3)}{\cong} \operatorname{Hom}_{\mathcal{H}}\big(\theta_{(\mu,\nu)}^{(\mu,\nu')}{}_\mu P^I, \theta_{(\lambda,\nu')}^{(\mu,\nu')}\theta_{(\lambda,\nu)}^{(\lambda,\nu')} X\big) \overset{(4)}{\cong} \operatorname{Hom}_{\mathcal{H}}\big({}_\mu P^J, \theta_{(\lambda,\nu')}^{(\mu,\nu')}\theta_{(\lambda,\nu)}^{(\lambda,\nu')} X\big)
\end{aligned}$$

$$\overset{(5)}{\cong} \mathbb{V}\theta_{(\lambda,\nu')}^{(\mu,\nu')}\theta_{(\lambda,\nu)}^{(\lambda,\nu')}X \overset{(6)}{\cong} \mathrm{res}_{\widehat{S}^\mu}^{\widehat{S}^\lambda} \mathbb{V}\theta_{(\lambda,\nu)}^{(\lambda,\nu')}X \overset{(7)}{\cong} \mathrm{res}_{\widehat{S}^\mu}^{\widehat{S}^\lambda} \mathrm{Hom}_{\mathcal{H}}\big({}_\lambda P^J, \theta_{(\lambda,\nu)}^{(\lambda,\nu')}X\big)$$

$$\overset{(8)}{\cong} \mathrm{res}_{\widehat{S}^\mu}^{\widehat{S}^\lambda} \mathrm{Hom}_{\mathcal{H}}\big(\theta_{(\lambda,\nu)}^{(\lambda,\nu')}{}_\lambda P^I, \theta_{(\lambda,\nu)}^{(\lambda,\nu')}X\big) \overset{(9)}{\cong} \mathrm{res}_{\widehat{S}^\mu}^{\widehat{S}^\lambda} \mathbb{V}\bigoplus_{i=1}^{|W_\nu|} X.$$

(The well-known regular situation is given by (6). The isomorphisms (1) to (3) and (9) follow from properties of translation functors; (4) and (8) are given by (2.7); and (5), (7) hold just by definition.) The resulting isomorphism restricts to a natural isomorphism $\mathbb{V}\theta_{(\lambda,\nu)}^{(\mu,\nu)}X \cong \mathrm{res}_{\widehat{S}^\mu}^{\widehat{S}^\lambda} \mathbb{V}X$ of $\widehat{S}^\mu \otimes \widehat{S}^\nu$-modules as follows: We consider X as a submodule of $\theta_{(\lambda,\nu')}^{(\lambda,\nu)}\theta_{(\lambda,\nu)}^{(\lambda,\nu')}X$ via the adjunction morphism. Then $f \in \mathbb{V}\theta_{(\lambda,\nu)}^{(\mu,\nu)}X = \mathrm{Hom}_{\mathcal{H}}({}_\mu P^I, \theta_{(\lambda,\nu)}^{(\mu,\nu)}X)$ corresponds via the canonical isomorphism (2) to $\theta_{(\mu,\nu)}^{(\mu,\nu')} f \in \mathrm{Hom}_{\mathcal{H}}(\theta_{(\mu,\nu)}^{(\mu,\nu')}{}_\mu P^I, \theta_{(\lambda,\nu')}^{(\mu,\nu')}\theta_{(\lambda,\nu)}^{(\lambda,\nu')}X)$. Again, we have a canonical isomorphism $\theta_{(\lambda,\nu')}^{(\mu,\nu')}\theta_{(\lambda,\nu)}^{(\lambda,\nu')} \cong \theta_{(\mu,\nu)}^{(\mu,\nu')}\theta_{(\lambda,\nu)}^{(\mu,\nu)}$. Following the sequence of isomorphisms, we get that f finally corresponds to $\theta_{(\lambda,\nu)}^{(\lambda,\nu')}\Phi(f) \in \mathrm{res}_{\widehat{S}^\mu}^{\widehat{S}^\lambda} \mathrm{Hom}_{\mathcal{H}}(\theta_{(\lambda,\nu)}^{(\lambda,\nu')}{}_\lambda P^I, \theta_{(\lambda,\nu)}^{(\lambda,\nu')}X)$, if $\Phi(f)$ denotes the image of f under the complete sequence. This proves statement (2.6). The isomorphism (2.4) can be proved in an analogous way. We omit the details. To prove the statements (2.3) and (2.5) it is sufficient to interchange the left and right $\mathcal{U}(\mathfrak{g})$-structure. $\square$

In [22], W. Soergel proved the faithfulness of $\mathbb{V}_{(0,0)}$ on projectives in ${}_0\mathcal{H}_0^n$. We will prove the corresponding statement for the blocks occurring in Theorem 2.2 and deduce the equivalence of categories. The first step is the following result

Theorem 2.4. *Let λ, μ be integral dominant weights and $I \overset{\lambda}{\triangleleft} \mathcal{Z}$. Let $X \in {}_\mu\mathcal{H}^I$ be projective. Then, the socle of X is a direct sum of modules of the form ${}_\lambda L_\mu$ (i.e. copies of the simple object with maximal Gelfand–Kirillov dimension).*

Proof. Note, that any simple object in ${}_\mu\mathcal{H}_\lambda$ is of the form $\mathcal{L}(M(\lambda), L(w\cdot\mu))$ for some $w \in W$. This object has maximal Gelfand–Kirillov dimension, if and only if so has $L(w\cdot\mu)$. The latter is exactly the case if $w\cdot\mu = w_o\cdot\mu$ (see [13, 10.12, 8.15, and 9.1]).

Take a filtration of $\mathcal{Z}$-modules

$$\mathcal{Z}/I = M_0 \supset M_1 \supset M_r \supset M_{r+1} = \{0\} \tag{2.8}$$

with maximal possible semisimple subquotients. The universal enveloping algebra is a free $\mathcal{Z}$-module, even a free left $\mathcal{Z}\otimes\mathcal{U}(\mathfrak{n}_-)$-module (see [16, Lemma 5.7]). Applying the (exact) functor $\mathcal{U}\otimes_{\mathcal{Z}} \bullet$ to the filtration above gives rise to a filtration of $\mathcal{U}\otimes_{\mathcal{Z}}\mathcal{Z}/I = \mathcal{U}/\mathcal{U}I$ with $\mathcal{M}_r := \mathcal{U}\otimes_{\mathcal{Z}} M_r \cong \bigoplus \mathcal{U}/\mathcal{U}(\ker\chi_\lambda)$, where the direct sum has $\dim_{\mathbb{C}}(M_r)$ many summands. Moreover, by construction, this submodule contains all elements annihilated by $\ker\chi_\lambda$. In particular, it contains the socle of $\mathcal{U}/\mathcal{U}I$. Obviously, $M \in {}_\lambda\mathcal{H}_\lambda^1$. This category is equivalent to a certain subcategory of $\mathcal{O}$ (via the functor $\mathcal{T}_{(\lambda,\lambda)}$ from the introduction) such that M corresponds to a direct sum of Verma modules $M(\lambda)$. Hence, the socle of M, and therefore also of $\mathcal{U} \otimes_{\mathcal{Z}} \mathcal{Z}/I$, consists only of simple modules with maximal Gelfand–Kirillov dimension. Since this property is still valid after tensoring with some finite-dimensional $\mathfrak{g}$-module E (see [13, 8.13]) and taking direct summands, the statement of the theorem follows by the previous description of the projective objects (Lemma 1.1). □

The following statement holds, in particular, for $X_1, X_2 \in {}_\mu\mathcal{H}^n_\lambda$.

Corollary 2.5. *Let λ, μ be integral dominant weights. Let $I \overset{\lambda}{\lhd} \mathcal{Z}$. Let $X_1, X_2 \in {}_\mu\mathcal{H}^I$ and X_2 be projective. Then $\mathbb{V} = \mathbb{V}_{(\mu,\lambda)}$ induces an inclusion*

$$\mathrm{Hom}_{\mathcal{H}}(X_1, X_2) \hookrightarrow \mathrm{Hom}_{\widehat{S}^\mu \otimes \widehat{S}^\lambda}(\mathbb{V}X_1, \mathbb{V}X_2).$$

Proof. The socle of any projective object contains only simple composition factors which are not annihilated by $\mathbb{V}$ and $\mathrm{im}\,\mathbb{V}f \cong \mathbb{V}\,\mathrm{im}\,f$ for any $f \in \mathrm{Hom}_{\mathcal{H}}(X_1, X_2)$. □

Lemma 2.6. *For λ, μ dominant integral weights and $I \overset{\lambda}{\lhd} \mathcal{Z}$. The following holds*:

(1) *There is an isomorphism of $\mathcal{Z}$-bimodules*

$$\mathbb{V}(\mathcal{U}/\mathcal{U}I) \xrightarrow{\sim} \mathcal{Z}/I$$

and via completion

$$\mathbb{V}(\mathcal{U}/\mathcal{U}I) \xrightarrow{\sim} \widehat{S}^\lambda/\hat{I}.$$

(2) $\mathbb{V}_\mu P^I \cong \widehat{S}^\mu \otimes_{\widehat{S}^W} \widehat{S}^\lambda/\hat{I}$. *In particular,* $\mathbb{V}_\mu P^n_\lambda \cong \widehat{S}^\mu \otimes_{\widehat{S}^W} \widehat{S}^\lambda/\mathfrak{m}^n_\lambda$ *for any* $n \in \mathbb{N}_+$.

Proof. The subquotients of the filtration (2.8) are isomorphic to $\mathcal{U} \otimes_{\mathcal{Z}} M_i/M_{i+1}$ and therefore contained in ${}_\lambda\mathcal{H}^1_\lambda$. Moreover, $\mathcal{T}_{(\lambda,\lambda)}(\mathcal{U} \otimes_{\mathcal{Z}} M_i/M_{i+1}) \cong \bigoplus_{j=1}^m M(\lambda)$, where $M(\lambda) \in \mathcal{O}$ is the Verma module of highest weight λ, the functor $\mathcal{T}_{(\lambda,\lambda)}$ denotes the BG-equivalence between ${}_\lambda\mathcal{H}^1_\lambda$ and a certain subcategory of $\mathcal{O}_\lambda$, and $m = \dim_{\mathbb{C}} M_i/M_{i+1}$. Therefore,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}}\big({}_\lambda P^I, \mathcal{U}/\mathcal{U}I\big) = [\mathcal{U}/\mathcal{U}I : {}_\lambda L_\lambda] = \sum_{i=0}^{n} \dim_{\mathbb{C}}(M_i/M_{i+1}) = \dim_{\mathbb{C}} \mathcal{Z}/I.$$

We claim that there is an inclusion of $\mathcal{Z}$-bimodules

$$\mathcal{Z}/I \hookrightarrow \mathrm{Hom}_{\mathcal{H}}\big({}_\lambda P^I, \mathcal{U}/\mathcal{U}I\big). \tag{2.9}$$

Since

$$\big[(\mathcal{U} \otimes_{\mathcal{Z}} \mathcal{Z}/I)/(\mathcal{U} \otimes_{\mathcal{Z}} \mathfrak{m}/I) : {}_\lambda L_\lambda\big] = 1,$$

there exists a (unique up to a scalar) morphism $f \in \mathrm{Hom}_{\mathcal{H}}\big({}_\lambda P^I, \mathcal{U}/\mathcal{U}I\big)$ such that $\mathrm{im}\, f \not\subseteq \mathcal{U} \otimes_{\mathcal{Z}} \mathfrak{m}/I$. On the other hand, $z.f = f.z \neq 0$ for any $z \in \mathcal{Z}/I$. That means, $z \mapsto z.f$ defines the required inclusion. The first part of the lemma follows since $\mathbb{V}\mathcal{U}/\mathcal{U}I \cong \mathrm{Hom}_{\mathfrak{g}}({}_\lambda P^I, \mathcal{U}/\mathcal{U}I)$.

To prove the second statement, we first show that there is an isomorphism of $\mathcal{U}$-bimodules

$$ {}_{-\rho}P^I \cong \theta_{(\lambda,\lambda)}^{(-\rho,\lambda)} \mathcal{U}/\mathcal{U}I. \tag{2.10}$$

If $N \in {}_{-\rho}\mathcal{H}^I$ then $\theta_{(-\rho,\lambda)}^{(\lambda,\lambda)} N \in {}_\lambda\mathcal{H}^I$. Therefore, $\theta_{(\lambda,\lambda)}^{(-\rho,\lambda)}\mathcal{U}/\mathcal{U}I \in {}_{-\rho}\mathcal{H}^I$ is projective by the projectivity of $\mathcal{U}/\mathcal{U}I \in {}_\lambda\mathcal{H}^I$ and by adjointness properties of translation functors. The category ${}_{-\rho}\mathcal{H}^I$ has up to isomorphism only one simple object, namely ${}_{-\rho}L_\lambda$, hence $\theta_{(\lambda,\lambda)}^{(-\rho,\lambda)}\mathcal{U}/\mathcal{U}I$ is a direct sum of copies of ${}_{-\rho}P^I$, the projective cover of ${}_{-\rho}L_\lambda$. On the other hand,

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}}\big(\theta_{(\lambda,\lambda)}^{(-\rho,\lambda)}\mathcal{U}/\mathcal{U}I, {}_{-\rho}L_\lambda\big) &= \dim_{\mathbb{C}} \mathrm{Hom}\big(\mathcal{U}/\mathcal{U}I, \theta_{(-\rho,\lambda)}^{(\lambda,\lambda)}{}_{-\rho}L_\lambda\big) \\ &= \big[\theta_{(-\rho,\lambda)}^{(\lambda,\lambda)}{}_{-\rho}L_\lambda : \mathcal{L}\big(M(\lambda), L(\lambda)\big)\big], \end{aligned} \tag{2.11}$$

where $\mathcal{L}(M(\lambda), L(\lambda))$ denotes the simple head of $\mathcal{U}/\mathcal{U}I$ (or of $\mathcal{U}/\mathcal{U}(\ker \chi_\lambda)$).

Since $\theta_{(-\rho,\lambda)}^{(\lambda,\lambda)}\,{}_{-\rho}L_\lambda \in {}_\lambda\mathcal{H}_\lambda^1$, the multiplicity (2.11) above is equal to

$$\big[\theta_{-\rho}^{\lambda} L(-\rho) : L(\lambda)\big] = \big[P(w_o \cdot \lambda) : L(\lambda)\big] = \big(P(w_o \cdot \lambda) : M(\lambda)\big) = 1.$$

(Here, $L(\mu) \in \mathcal{O}$ denotes the simple module with highest weight $\mu \in \mathfrak{h}^*$ and projective cover $P(\mu)$.) This proves the isomorphism (2.10). Combining it with Lemma 2.3 and the first part of the lemma, we get the following isomorphisms:

$$\begin{aligned} \mathbb{V}_\lambda P^I &\cong \mathbb{V}\theta_{(-\rho,\lambda)}^{(\mu,\lambda)}\theta_{(\lambda,\lambda)}^{(-\rho,\lambda)}\mathcal{U}/\mathcal{U}I \cong \big(\widehat{S}^\mu \otimes \widehat{S}^\lambda\big) \otimes_{(\widehat{S}^{-\rho}\otimes\widehat{S}^\lambda)} \mathrm{res}_{\widehat{S}^{-\rho}\otimes\widehat{S}^\lambda}^{\widehat{S}^\lambda\otimes\widehat{S}^\lambda} \mathbb{V}\mathcal{U}/\mathcal{U}I \\ &\cong \widehat{S}^\mu \otimes_{\widehat{S}^W} \widehat{S}^\lambda/\hat{I}. \end{aligned}$$

This proves the lemma. □

Proof of Theorem 2.2. In the situation of the theorem, there is up to isomorphism only one simple object, namely ${}_\mu L_\lambda$, in ${}_\mu\mathcal{H}_\lambda^n$. Its projective cover is therefore a minimal projective generator. By Corollary 2.5, it is sufficient to show that $\dim_{\mathbb{C}} \mathrm{End}_{\mathcal{H}}({}_\mu P_\lambda^n) = \dim_{\mathbb{C}} \mathrm{End}_{\widehat{S}^\mu\otimes\widehat{S}^\lambda}(\mathbb{V}_\mu P_\lambda^n)$.

The definition of $\mathbb{V}$ and Lemma 2.6 give the equalities

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{End}_{\mathcal{H}}\big({}_\mu P_\lambda^n\big) = \dim_{\mathbb{C}}\big(\mathbb{V}_\mu P_\lambda^n\big) &= \dim_{\mathbb{C}}\big(\widehat{S}^\mu \otimes_{\widehat{S}^W} \widehat{S}^\lambda/\mathfrak{m}_\lambda^n\big) \\ &= |W/W_\mu| \cdot \dim_{\mathbb{C}}\big(\widehat{S}^\lambda/(\mathfrak{m}_\lambda)^n\big), \end{aligned}$$

since $\widehat{S}^\lambda$ is a free $\widehat{S}^W$ module of rank $|W/W_\mu|$. On the other hand,

$$\begin{aligned}\mathrm{End}_{\widehat{S}^{\mu}\otimes\widehat{S}^{\lambda}}\big(\widehat{S}^{\mu}\otimes_{\widehat{S^W}}\widehat{S}^{\lambda}/(\mathfrak{m}_\lambda)^n\big) &\cong \mathrm{Hom}_{\widehat{S}^W\otimes\widehat{S}^{\lambda}}\big(\widehat{S}^{\lambda},\widehat{S}^{\mu}\otimes_{\widehat{S}^W}\widehat{S}^{\lambda}/(\mathfrak{m}_\lambda)^n\big)\\ &\cong \mathrm{Hom}_{\mathrm{mod}-\widehat{S}^{\lambda}}\big(\widehat{S}^{\lambda},\widehat{S}^{\mu}\otimes_{\widehat{S}^W}\widehat{S}^{\lambda}/(\mathfrak{m}_\lambda)^n\big)\\ &\cong \bigoplus_{i=1}^{|W/W_\mu|}\mathrm{End}_{\widehat{S}^{\lambda}}\big(\widehat{S}^{\lambda}/(\mathfrak{m}_\lambda)^n\big).\end{aligned}$$

The theorem follows. □

3. The Bernstein–Gelfand equivalence: an obvious generalisation?

Recall the equivalence of categories (0.2) with its generalisation to the singular case. In this section we deal with the question whether it can directly be generalised to an equivalence of categories between ${}_\lambda\mathcal{H}_\mu^n$ and a subcategory of projectively presentable objects in the 'thick' category $\mathcal{O}^n$. This is a full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules. A $\mathcal{U}(\mathfrak{g})$-module M is an object of $\mathcal{O}^n$ if it satisfies the following conditions:

(1) M is a finitely generated $\mathcal{U}(\mathfrak{g})$-module;
(2) $M=\bigoplus_{\lambda\in\mathfrak{h}^*}M_\lambda^n$, where $M_\lambda^n=\{m\in M\mid (h-\lambda(h))^n m=0\}$; and
(3) M is locally $\mathcal{U}(\mathfrak{b})$-finite, i.e. $\dim_{\mathbb{C}}\mathcal{U}(\mathfrak{b})m<\infty$ for all $m\in M$.

Note that for $n=1$ this is just the category $\mathcal{O}$ of [5]. The action of the centre gives a decomposition

$$\mathcal{O}^n=\bigoplus_\mu \mathcal{O}_\mu^n,$$

where μ runs through a system of dominant orbit representatives of $\mathfrak{h}^*/W\cdot$. This is in bijection to the maximal ideals of the centre of $\mathcal{U}$ by the Harish-Chandra isomorphism. (For more details, see [20,22]). Let $L_{w_0}\in\mathcal{O}_0^n$ be the simple Verma module (with highest weight $w_o\cdot 0$). Let $P_{w_0}^n$ be its projective cover in $\mathcal{O}_0^n$. For $\lambda\in\mathfrak{h}^*$ and $n\in\mathbb{Z}_{>0}$, we define the 'deformed' Verma module

$$M^n(\lambda)=\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}S/(\ker\lambda)^n,$$

where S becomes a $\mathcal{U}(\mathfrak{b})$-module via the canonical surjection $\mathfrak{b}\twoheadrightarrow\mathfrak{h}$. Note that for $n=1$ this is the usual Verma module with highest weight λ. For λ dominant, $M^n(\lambda)\in\mathcal{O}_\lambda^n$.

The following theorem is due to Soergel.

Theorem 3.1. *There are isomorphisms of algebras*

$$\mathrm{End}_{\mathcal{H}}\big({}_0P_0^n\big)\cong\mathrm{End}_{\mathcal{O}^n}\big(P_{w_0}^n\big)\cong S\otimes_{S^W}S/(\mathfrak{h})^n.$$

Proof. The equivalence of categories [20, Proposition 1] sends ${}_0P_0^n$ to $P_{w_0}^n$ and gives the first isomorphism. For the second isomorphism we consider $P_{w_0}^n$ as the $(\mathfrak{h})^n$-specialisation of the 'deformation $\mathbf{P}_0$ of the antidominant projective $P(w_0 \cdot 0) \in \mathcal{O}_0$' [22, Theorem 7]. Since specialisation is compatible with morphism spaces in the sense of [22, Theorem 5], the Endomorphism Theorem for $\mathbf{P}_0$ [22, Theorem 9] implies

$$\operatorname{End}_{\mathcal{O}^n}\left(P_{w_0}^n\right) \cong \mathbf{T} \otimes_{\mathbf{T}^W} \mathbf{T}/(\mathfrak{h})^n,$$

where $\mathbf{T}$ is the localisation of T at the maximal ideal generated by $\mathfrak{h}$. Let $f, g \in S$ and $g(0) \neq 0$. Then

$$\frac{f}{g} = \frac{f \prod_{w \in W,\, w \neq e} g^w}{\prod_{w \in W} g^w},$$

with invariant denominator, hence $\mathbf{T} \otimes_{\mathbf{T}^W} \mathbf{T}/(\mathfrak{h})^n = S \otimes_{S^W} S/(\mathfrak{h})^n$. This proves the theorem. □

A module $M \in \mathcal{O}^n$ is called $P_{w_0}^n$*-presentable*, if there is an exact sequence of the form $P_1 \to P_2 \to M \to 0$, where P_1 and P_2 are finite direct sums of $P_{w_0}^n$. In [5], it was proved that the functor $\bullet \otimes_{\mathcal{U}} M(-\rho)$ defines an equivalence of categories between ${}_0\mathcal{H}_{-\rho}^1$ and the full subcategory of $\mathcal{O}_0$ given by $P_{w_0}^1$-presentable objects. The following negative result seems to be important enough to state it.

Corollary 3.2. *In general, the category ${}_0\mathcal{H}_{-\rho}^n$ is not equivalent to the full subcategory $\mathcal{P}^n$ of $\mathcal{O}_0^n$ defined by the $P_{w_0}^n$-presentable objects.*

Proof. By Theorem 3.1, $\operatorname{End}_{\mathcal{O}^n}(P_{w_0}^n) \cong S \otimes_{S^W} S/(\mathfrak{h})^n$, hence the subcategory $\mathcal{P}^n$ in question is equivalent to the category of finitely generated $S \otimes_{S^W} S/(\mathfrak{h})^n$-modules. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $n \geqslant 2$. Then $S \otimes_{S^W} S/(\mathfrak{h})^n \cong \mathbb{C}[x] \otimes_{(x^2)} \mathbb{C}[x]/(x^n)$. It is easy to check that the map $x \mapsto 1 \otimes x$ and $y \mapsto x \otimes 1$ defines an isomorphism $\mathbb{C}[x,y]/\langle x^n, x^2+y^2\rangle \cong \mathbb{C}[x] \otimes_{(x^2)} \mathbb{C}[x]/(x^n)$. We claim that there exists an isomorphism of graded algebras

$$\mathbb{C}[x,y]\big/\big\langle x^n, x^2+y^2\big\rangle \cong \mathbb{C}[x,y]\big/\big(\big(\mathbb{C}[x,y]^+\big)^{D_n}\big), \tag{3.1}$$

where D_n denotes the dihedral group of order $2n$. In fact, the group D_n is generated by the maps

$$(x,y) \mapsto \big(\cos(2\pi/n)x + \sin(2\pi/n)y, \cos(2\pi/n)y - \sin(2\pi/n)x\big) \quad \text{and}$$
$$(x,y) \mapsto (x,-y).$$

Direct calculations show that $x^2 + y^2$ is an invariant polynomial. A second generator of the invariants can be therefore chosen homogeneous of degree n (see [15, 17.4]), where y occurs only with even exponents. This implies that x^n is contained in the ideal generated by the invariants. Comparing the dimensions yields the required isomorphism. In particular $\mathrm{Ext}^\bullet_{S\otimes_{S^W} S/(\mathfrak{h})^n} \cong \mathbb{C}[x, y]$ as a graded vector space (see proof of Theorem 1.3). On the other hand, $\widehat{S} \otimes_{\widehat{S}^W} \widehat{S}^W/\mathfrak{m}^n_{-\rho} \cong \mathbb{C}[x]/(x^{2n})$, hence $\mathrm{Ext}^\bullet_{{}_0\mathcal{H}^n_{-\rho}}({}_0L_{-\rho}, {}_0L_{-\rho}) \cong \mathrm{Ext}^\bullet_{\mathbb{C}[x]/(x^{2n})}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[x]$ as graded vector spaces. In particular, the categories in question are not equivalent. □

4. The combinatorics of Harish-Chandra bimodules

In this section we prove the following general result.

Theorem 4.1. *Let $\lambda, \mu \in \mathfrak{h}^*$ be dominant and integral and $I \overset{\lambda}{\lhd} \mathcal{Z}$. Let $P, Q \in {}_\mu\mathcal{H}^I$ be projective. Then $\mathbb{V}$ induces an isomorphism*

$$\alpha_{Q,P} : \mathrm{Hom}_{\mathcal{H}}(Q, P) \cong \mathrm{Hom}_{\widehat{S}^\mu \otimes \widehat{S}^\lambda}(\mathbb{V}Q, \mathbb{V}P).$$

We start with some preparatory lemmata.

Lemma 4.2. *Let $\lambda \in \mathfrak{h}^*$ be an integral and dominant weight. Let $I \overset{\lambda}{\lhd} \mathcal{Z}$ and let $P \in {}_\mu\mathcal{H}^I$ be projective. There exists an exact sequence of the form*

$$0 \longrightarrow P \longrightarrow \bigoplus_{i=1}^{m_1} {}_\mu P^I \longrightarrow \bigoplus_{i=1}^{m_2} {}_\mu P^I$$

for some $m_1, m_2 \in \mathbb{N}$.

Proof. Let first $\mu = \lambda$. Let $L \in {}_\lambda\mathcal{H}^I$ be simple of non-maximal Gelfand–Kirillov dimension. We claim that

$$\mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}(L, \mathcal{U}/\mathcal{U}I) = 0. \tag{4.1}$$

For any simple reflection s, the adjunction morphism gives (Theorem 2.4) a short exact sequence of the form

$$\mathcal{U}/\mathcal{U}I \hookrightarrow \theta_s^l \mathcal{U}/\mathcal{U}I \twoheadrightarrow K_s \tag{4.2}$$

for some $K_s \in {}_\lambda\mathcal{H}^I$. The bimodule $\mathcal{U}/\mathcal{U}I$ has a filtration with subquotients isomorphic to $\mathcal{U}/\mathcal{U}\ker\chi_\lambda$. Since $\mathcal{T}_{(\lambda,\lambda)}\mathcal{U}/\mathcal{U}\ker\chi_\lambda \cong M(\lambda)$ and $\mathcal{T}_{(\lambda,\lambda)}\theta_s^l \cong \theta_s\mathcal{T}_{(\lambda,\lambda)}$, the exactness of θ_s^l implies $\mathrm{Hom}_{\mathcal{H}}(L, K_s) = 0$. (Here, θ_s denotes the translation through the wall in category $\mathcal{O}$.) We choose s such that $\theta_s^l L = 0$. From (4.2) we get an exact sequence

$$\mathrm{Hom}_{\mathcal{H}}(L, K_s) \to \mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}(L, \mathcal{U}/\mathcal{U}I) \to \mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}\big(L, \theta_s^l\mathcal{U}/\mathcal{U}I\big),$$

where the outer terms are zero, hence (4.1) holds. Set $\mathcal{F}=\theta_{(-\rho,\lambda)}^{(\lambda,\lambda)}\theta_{(\lambda,\lambda)}^{(-\rho,\lambda)}$. The adjunction morphism $\mathcal{U}/\mathcal{U}I\hookrightarrow\mathcal{F}\mathcal{U}/\mathcal{U}I$ is injective (by Theorem 2.4). Let K be the cokernel. From the exactness of

$$0=\mathrm{Hom}_{\mathcal{H}}(L,\mathcal{F}\mathcal{U}/\mathcal{U}I)\to\mathrm{Hom}_{\mathcal{H}}(L,K)\to\mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}(L,\mathcal{U}/\mathcal{U}I)$$

it follows $\mathrm{Hom}_{\mathcal{H}}(L,K)=0$. Hence $K\hookrightarrow\mathcal{F}K$ via the adjunction morphism. By adjunction, $\mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}(\mathcal{F}K,L)=0$ if $L\neq{}_\lambda L_\lambda$. For $L={}_\lambda L_\lambda$, we have an exact sequence

$$\begin{aligned}0&\to\mathrm{Hom}_{\mathcal{H}}(\mathcal{F}K,L)\to\mathrm{Hom}_{\mathcal{H}}(\mathcal{F}^2\mathcal{U}/\mathcal{U}I,L)\to\mathrm{Hom}_{\mathcal{H}}(\mathcal{F}\mathcal{U}/\mathcal{U}I,L)\\&\to\mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}(\mathcal{F}K,L)\to\mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}(\mathcal{F}^2\mathcal{U}/\mathcal{U}I,L)\to\cdots\end{aligned}$$

The last term is zero, since $\mathcal{U}/\mathcal{U}I\in{}_\lambda\mathcal{H}^I$ is projective; $\mathrm{Hom}_{\mathcal{H}}(\mathcal{F}\mathcal{U}/\mathcal{U}I,L)=\mathbb{C}$ (see (2.10) and Lemma 2.3) and $\dim\mathrm{Hom}_{\mathcal{H}}(\mathcal{F}^2\mathcal{U}/\mathcal{U}I,L)=|W_\lambda|$ [5, 4.2c]. On the other hand, $\mathcal{U}/\mathcal{U}I$ surjects onto $\mathcal{U}/(\ker\chi_\lambda)\mathcal{U}$, hence $\mathcal{F}(K)$ surjects onto

$$Q:=\mathcal{F}\big(\mathrm{coker}\big((\mathcal{U}/\ker\chi_\lambda)\to\mathcal{F}(\mathcal{U}/\ker\chi_\lambda)\big)\big),$$

i.e. $\mathcal{F}$ applied to the cokernel of the adjunction morphism. Standard arguments in category $\mathcal{O}$ give $Q\cong\bigoplus_{i=1}^{|W_\lambda|-1}{}_\lambda P^1_\lambda$. Therefore, $\dim\mathrm{Hom}_{\mathcal{H}}(\mathcal{F}K,L)\geqslant|W_\lambda|-1$ and hence $\mathrm{Ext}^1_{{}_\lambda\mathcal{H}^I}(\mathcal{F}K,L)=0$ for any simple object $L\in\mathcal{H}$. In particular, $\mathcal{F}K$ is projective and by adjointness properties of $\mathcal{F}$ it follows that $\mathcal{F}K$ is a direct sum of copies of ${}_\lambda P^I$. Altogether, we get the existence of an exact sequence as in the lemma in the case $\mu=\lambda$ and $P=\mathcal{U}/\mathcal{U}I$. The general statement follows from Theorem 1.1 using translation functors. □

Lemma 4.3. *If P, $Q\in{}_\mu\mathcal{H}^I$ are projective and $P={}_\mu P^I$, then $\alpha_{Q,P}$ is an isomorphism.*

Proof. By Corollary 2.5, we only have to compare the dimensions. Since $\theta_{(\mu,\lambda)}^{(-\rho,\lambda)}Q\in{}_{-\rho}\mathcal{H}^I$ is projective, $\theta_{(\mu,\lambda)}^{(-\rho,\lambda)}Q\cong\bigoplus_{i=1}^r{}_{-\rho}P^I$ where

$$r=\frac{[Q:{}_\lambda L_\lambda]}{[{}_{-\rho}P^I:{}_{-\rho}L_\lambda]}=\frac{[Q:{}_\lambda L_\lambda]}{[\mathcal{U}/\mathcal{U}I:{}_\lambda L_\lambda]}=\frac{[Q:{}_\lambda L_\lambda]}{\dim\mathcal{Z}/I}.$$

Note that we used Lemma 2.6 for the last equality. On the other hand,

$$\mathrm{Hom}_{\widehat{S}^\mu\otimes\widehat{S}^\lambda}\big(\mathbb{V}Q,\mathbb{V}_\lambda P^I\big)\cong\mathrm{Hom}_{\widehat{S}^\mu\otimes\widehat{S}^\lambda}\big(\mathbb{V}Q,\widehat{S}^\mu\otimes_{\widehat{S}^W}\widehat{S}^\lambda/\hat{I}\big)\cong\mathrm{Hom}_{\widehat{S}^W\otimes\widehat{S}^\lambda}\big(\mathbb{V}Q,\widehat{S}^\lambda/\hat{I}\big).$$

Since Q is projective, Theorem 1.1 and Lemma 2.3 imply that $\mathbb{V}Q$ is a direct summand of some $\mathcal{G}\,\mathbb{V}\mathcal{U}/\mathcal{U}I$, where $\mathcal{G}$ is given by a composition of induction and restriction functors as in Lemma 2.3. In particular, $\operatorname{Hom}_{\widehat{S}^W \otimes \widehat{S}^\lambda}(\mathbb{V}Q, \widehat{S}^\lambda/\hat{I}) = \operatorname{Hom}_{\text{mod-}\widehat{S}^\lambda}(\mathbb{V}Q, \widehat{S}^\lambda/\hat{I})$. Moreover, $\mathbb{V}Q$ is a direct summand of a free right $\widehat{S}^\lambda/\hat{I}$-module, hence itself projective, and therefore also free (since $\widehat{S}^\lambda$ is a local ring). The rank of $\mathbb{V}Q$ is equal to

$$\frac{\dim \mathbb{V}Q}{\dim \widehat{S}^\lambda/\hat{I}}.$$

Comparison with the formula above gives the desired result. □

Proof of Theorem 4.1. Let P, Q be as in the theorem. Let $P \overset{j}{\hookrightarrow} P_1 \to P_2$ be an exact sequence as in Lemma 4.2. This provides a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Hom}_{\mathcal{H}}(Q, P) & \longrightarrow & \operatorname{Hom}_{\mathcal{H}}(Q, P_1) & \longrightarrow & \operatorname{Hom}_{\mathcal{H}}(Q, P_2) \\
& \downarrow{\scriptstyle \alpha_{Q,P}} & & \downarrow{\scriptstyle \alpha_{Q,P_1}} & & \downarrow{\scriptstyle \alpha_{Q,P_2}} \\
& \operatorname{Hom}_{\widehat{S}^\mu \otimes \widehat{S}^\lambda}(\mathbb{V}Q, \mathbb{V}P) & \xrightarrow{\phi := \mathbb{V}j \circ} & \operatorname{Hom}_{\widehat{S}^\mu \otimes \widehat{S}^\lambda}(\mathbb{V}Q, \mathbb{V}P_1) & \longrightarrow & \operatorname{Hom}_{\widehat{S}^\mu \otimes \widehat{S}^\lambda}(\mathbb{V}Q, \mathbb{V}P_2).
\end{array}$$

Since j is injective and $\mathbb{V}$ is exact, $\mathbb{V}j$ is injective and hence ϕ as well. The Lemma 4.3 implies the theorem for $P, Q \in {}_\mu\mathcal{H}^I$ projective. For Q arbitrary one takes a projective resolution. The full statement follows then easily using the five lemma. □

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