# SEMINAR: SEMISIMPLE AND NILPOTENT ORBITS IN ALGEBRAIC GROUPS

## 1. Nilpotent and semisimple elements

- Jordan decomposition
- Def. ss./reductive Lie algebra
- Spectral decomposition with special example weight space decomposition
- Def. Killing form
- A Lie algebra is semisimple if and only if the Killing form is non-degenerate
- Example: $\mathfrak{sl}(2,\mathbb{C})$,
- Explain the "Problem" and "Refined Problem" about semisimple and nilpotent elements
- Examples: $\mathfrak{sl}(2,\mathbb{C})$, remarks on $\mathfrak{sl}(n,\mathbb{C})$.

Reference: Collinwood/Mc Govern: pages 1-7,
for the proofs, parts from
Humphreys: Introduction into semisimple Lie algebras
Steinberg: Conjugacy classes in Algebraic groups
are needed (find the corresponding statement there yourself).

## 2. Adjoint group, (co)adjoint orbit

- Definition of adjoint group: 2 constructions (connected complex Lie group with Lie algebra $ad_{\mathfrak{g}}$ or as image of $Ad$)
- Explain the "tower" of groups with the same Lie algebra (with $G_{sc}$ at the top and $G_{ad}$ at the bottom
- Explain the special example for $\mathfrak{sl}_2$.
- Def. Adjoint orbit
- Adjoint orbits as homogeneous spaces
- Dimension of adjoint orbits
- Two ways to construct a coadjoint orbit ("Lazy" and "Sophisticated")

Basic reference: Collinwood/Mc Govern: pages 8-14,
Humphreys: Introduction into semisimple Lie algebras
Conjugacy classes in Algebraic groups are needed (find the corresponding statement there yourself).

## 3. Orbits as symplectic manifolds

- Recall briefly the two approaches to coadjoint orbits
- Definition of a symplectic vector space
- symplectic manifold
- Darboux coordinates
- isotropic subspaces, Lagrangian subspaces
- Main theorem: Coadjoint orbits are symplectic manifolds (idea of the proof only)

References:
Collinwood/Mc Govern: Section 1.4 (Try to work through the proof of Theorem 1.4.7 with the reference given in the book. Give at least an idea of the proof. State Theorem 1.4.9 without proof)
Berndt: An Introduction to Symplectic geometry Section 1.1, Section 2.1 and 2.2

## 4. Semisimple Orbits

- definition of Cartan subalgebras
- definition regular semisimple element
- Main theorem: Cartan subalgebras in a reductive Lie algebra over $\mathbb{C}$ are conjugate and of dimension rank of $\mathfrak{g}$.
- Cartan algebras in a reductive Lie algebra are all obtained as centralisers.
- Regular semisimple elements form a connected Zariski-open and dense set.
- Example: $\mathfrak{sl}(2, \mathbb{C})$.
- definition of regular elements
- Examples in $\mathfrak{sl}(n, \mathbb{C})$.

## 5. Classification of semisimple orbits

- Main result: semisimple orbits are in bijection to $W$-orbits on $\mathfrak{h}$.
- Explain Lemma 2.2.3. and the W-action on $\mathfrak{h}$ in a special example
- Example: $\mathfrak{sl}(4, \mathbb{C})$.

Reference: Collinwood/Mc Govern: Section 2.2 (but use Humphreys' book and the references given in this Section to fill in the details).

## 6. Basic topology of semisimple Orbits

- Borel/Harish-Chandra theorem: the adjoint orbit of an element $X$ in a reductive Lie algebra is closed if and only if $X$ is semisimple.
- In the case above, the orbit is also simply connected.
  Reference: Collinwood/Mc Govern: Section 2.3 and the references given there.

## 7. Dynkin-Kostant classification I (statement and type A) and Jacobson-Morosov

- Statement of the Dynkin-Kostant classification
- Type A: nilpotent orbits are in bijection to partitions
- In a semisimple Lie algebra, every nilpotent element lies in the nilradical of a Borel
- Jacobson-Morosov Theorem

References:
Collinwood/Mc Govern: Introduction to Section 3, Proposition 3.1.7 with proof, Lemma 3.2.1 (give only the ideas of the proof), Lemma 3.2.2 with proof (illustrate this with an example!) Lemma 3.2.6 without proof, Corollary 3.2.7 (important!), Theorem 3.3.1.

## 8. Dynkin-Kostant classification II

- Explain the strategy to the classification of nilpotent orbits
- Explain the relation between the set of $G_{ad}$ conjugacy classes of standard triples in $\mathfrak{g}$ and nilpotent orbits (without giving details)
- Explain the steps of the classification theorem of nilpotent orbits without giving proofs. Make sure that the statements of the theorems used are clear.
- Prove Malcev's Theorem (the proof is technical, so illustrate each step by a concrete example!)
- Give a handout with the main ideas of Kostant's Theorem.

References:
Collinwood/Mc Govern: Section 3.2, Theorem 3.2.10 (Classification Step I), Theorem 3.4.12 with proof, create a handout for the main steps in the proof of Theorem 3.4.10.

## 9. Weighted Dynkin diagrams

- The definition of a weighted Dynkin diagram
- Definition: principal nilpotent orbit in terms of a weighted Dynkin diagram
- Kostant's theorem: The weighted Dynkin diagram is an invariant of a nilpotent orbit.
- weighted Dynkin diagrams in type A
- Explain the notion of a parabolic subalgebra
- Give an overview on the partition type classification and the weighted Dynkin diagrams in the classical types

References:
Collinwood/Mc Govern: Section 3.5 and Section 3.6 in detail, Lemma 3.8.1 without proof, Lemma 3.8.4 Table on page 54.

## 10. Principal and subregular orbit

- Explain the partial ordering on the set of nilpotent orbits.
- Explain the notion of principal orbit and prove the Siebenthal-Dynkin-Kostant Theorem.
- Prove Steinberg's theorem and explain the notion of a subregular orbit.

References:
Collinwood/Mc Govern: Section 4.1 and Section 4.2

## 11. The minimal nilpotent orbit (+ connections with group cohomology

- Explain the notion of minimal nilpotent orbit
- Main Result: Existence of a minimal nilpotent orbit and dimension formulas (for simple Lie algebras)
- Example $\mathfrak{sl}(4, \mathbb{C})$.

  References:
  Collinwood/Mc Govern: Section 4.3. Section 4.4

## 12. Topology of nilpotent orbits

– Partial ordering on partitions
– Inclusion ordering of nilpotent orbits in classical groups
possible additional topics
– The fundamental group of a nilptent orbit
– The component group
References:
Collinwood/Mc Govern: Section 6.2

## 13. Quantisation of Slodowy slices I: Poisson structures and coadjoint orbits

– Definition Poisson bracket
– Definition of a Poisson manifold
– Symplectic leaves
– Example: $\mathfrak{g}^*$ with the symplectic leaves being the coadjoint orbits.

## 14. Quantisation of Slodowy slices II

– This will be a more advanced talk on the Theorem 4.1 in the paper

Gan, Ginzburg: Quantization of Slodowy slices

http://arxiv.org/PS_cache/math/pdf/0105/0105225v2.pdf