COMPLETIONS OF GROTHENDIECK GROUPS

PRAMOD N. ACHAR AND CATHARINA STROPPEL

Abstract. For a certain class of abelian categories, we show how to make sense of the "Euler characteristic" of an infinite projective resolution (or, more generally, certain chain complexes that are only bounded above), by passing to a suitable completion of the Grothendieck group. We also show that right-exact functors (or their left-derived functors) induce continuous homomorphisms of these completed Grothendieck groups, and we discuss examples and applications coming from categorification.

1. Introduction

Let $\mathcal{A}$ be a noetherian and artinian abelian category with enough projectives and $D^{\mathrm{b}}(\mathcal{A})$ its bounded derived category. The inclusion $\mathcal{A} \to D^{\mathrm{b}}(\mathcal{A})$ gives rise to a natural isomorphism of Grothendieck groups

$$K(\mathcal{A}) \xrightarrow{\sim} K(D^{\mathrm{b}}(\mathcal{A})). \tag{1.1}$$

When $\mathcal{A}$ has finite cohomological dimension, $K(\mathcal{A})$ captures a great deal of information about "derived" phenomena. For instance, for any $X \in \mathcal{A}$, we have

$$[X] = \sum (-1)^i [P^i], \qquad \text{where } P^\bullet \to X \text{ is a projective resolution.} \tag{1.2}$$

If $\mathcal{B}$ is another such category, then for any right-exact functor $F : \mathcal{A} \to \mathcal{B}$, the derived functor $\mathcal{L}F$ induces a group homomorphism

$$[\mathcal{L}F] : K(\mathcal{A}) \to K(\mathcal{B}). \tag{1.3}$$

On the other hand, if $\mathcal{A}$ has infinite projective dimension, we should replace $D^{\mathrm{b}}(\mathcal{A})$ by $D^-(\mathcal{A})$, the bounded above derived category, but then its Grothendieck group cannot see many derived phenomena. The natural map $K(\mathcal{A}) \to K(D^-(\mathcal{A}))$ may be the zero map (indeed, one may have $K(D^-(\mathcal{A})) = 0$; see [13]); the infinite sum in (1.2) does not make sense; and for (1.3), $[\mathcal{L}F]$ is only defined on the proper subgroup of $K(\mathcal{A})$ spanned by projectives.

However, when $\mathcal{A}$ and $\mathcal{B}$ are mixed categories with a Tate twist, a version of the statements (1.1), (1.2), and (1.3) can be recovered.

We explain in this note how to replace $D^-(\mathcal{A})$ by a certain subcategory $D^\nabla(\mathcal{A})$ that is still large enough for derived functors, but small enough that (1.2) holds after passing to a suitable completion. Certain infinite sums like (1.2) converge in this completion, and derived functors induce *continuous* homomorphisms of the completed Grothendieck groups.

More precisely, in this setting, the Grothendieck group $K(\mathcal{A})$ is naturally a module over the ring $\mathfrak{R} = \mathbb{Z}[q, q^{-1}]$. It admits a completion $\hat{K}(\mathcal{A})$ that is a module over $\hat{\mathfrak{R}} = \mathbb{Z}[[q]][q^{-1}]$. The main results of the paper are summarized below. (Further definitions and notation are given in Section 2.)

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Theorem 1.1. *Let $\mathcal{A}$ be a noetherian and artinian mixed abelian category with enough projectives and a Tate twist.*

(1) *$K(D^\nabla(\mathcal{A}))$ is a complete topological $\mathfrak{R}$-module. Moreover, the natural map $K(\mathcal{A}) \to K(D^\nabla(\mathcal{A}))$ is injective and induces an isomorphism*

$$\hat{K}(\mathcal{A}) \xrightarrow{\sim} K(D^\nabla(\mathcal{A})).$$

(2) *Every object $X \in D^\nabla(\mathcal{A})$ admits a projective resolution $P^\bullet$ with asymptotically decreasing weights. In $\hat{K}(\mathcal{A})$, we have convergent series*

$$[X] = \sum_{i\in\mathbb{Z}} (-1)^i [H^i(X)] = \sum_{i\in\mathbb{Z}} (-1)^i [P^i].$$

(3) *Both $\mathrm{Irr}(\mathcal{A}_0)$ and $\mathrm{Proj}(\mathcal{A})_0$ span dense free $\hat{\mathfrak{R}}$-submodules of $\hat{K}(\mathcal{A})$. If those sets are finite, they each give an $\hat{\mathfrak{R}}$-basis for $\hat{K}(\mathcal{A})$.*

(4) *Let $\mathcal{B}$ be another finite-length mixed category with a Tate twist, and let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact functor that commutes with Tate twist. If F has finite weight amplitude, then $\mathcal{L}F$ induces a continuous homomorphism of $\hat{\mathfrak{R}}$-modules*

$$[\mathcal{L}F] : \hat{K}(\mathcal{A}) \to \hat{K}(\mathcal{B}).$$

The idea of completing Grothendieck groups arises from the concept of *categorification*, see e.g. [9, 12]. There, $\mathbb{Z}[q, q^{-1}]$-modules get realized as Grothendieck groups $K(\mathcal{A})$ of appropriately chosen graded categories $\mathcal{A}$. The action of q and q^{-1} arises from shifting the grading (up and down). Often the $\mathbb{Z}[q, q^{-1}]$-modules in question come along with standard and canonical bases which then correspond to distinguished bases of $K(\mathcal{A})$. So far, representation theorists focused on the cases where the entries of the transformation matrices between the different bases were elements of $\mathbb{Z}[q, q^{-1}]$. These numbers are then usually interpreted as Jordan–Hölder multiplicities or graded decomposition numbers. However, the theory of Lusztig's canonical bases or Kashiwara's crystal bases gives plenty of examples where the entries of the transformation matrix are contained only in a completion, $\mathbb{Z}[[q]][q^{-1}]$ of $\mathbb{Z}[q, q^{-1}]$. Theorem 1.1 provides a possible categorical setup to handle such situations and could be viewed as the abstract context for categorifications of, for instance, modules for quantum groups. Although this paper focuses on the abstract setup, there are already concrete examples known, for instance in the context of categorification of Reshetikhin–Turaev–Viro invariants of links and 3-manifolds, see [10].

Following some set-up in Section 2, the main theorem will be proved in Sections 3 and 4. Some examples and applications are indicated in Section 5.

2. Notation and Definitions

2.1. **Mixed abelian categories.** All abelian categories will be assumed to be *finite-length categories* (i.e., noetherian and artinian) and to be skeletally small. For an abelian category $\mathcal{A}$, let $\mathrm{Irr}(\mathcal{A})$ denote the set of isomorphism classes of simple objects. In this setting, the Grothendieck group $K(\mathcal{A})$ is a free abelian group on the set $\mathrm{Irr}(\mathcal{A})$. Recall that $K(\mathcal{A}) = F(\mathcal{A})/R(\mathcal{A})$, where $F(\mathcal{A})$ is the free abelian group on isomorphism classes $[M]$ of objects $M \in \mathcal{A}$, and $R(\mathcal{A})$ is the ideal generated by the expressions $[A]-[C]+[B]$ whenever there is a short exact sequence

of the form $0 \to A \to C \to B \to 0$. So the above claim follows from the existence and uniqueness of a Jordan–Hölder series and states that

$$K(\mathcal{A}) = \mathbb{Z}[\mathrm{Irr}(\mathcal{A})]. \tag{2.1}$$

Recall that $\mathcal{A}$ is said to be a *mixed* category if there is a function $\mathrm{wt} : \mathrm{Irr}(\mathcal{A}) \to \mathbb{Z}$, called the *weight function* such that $\mathrm{Ext}^1(L, L') = 0$ if $\mathrm{wt}([L]) \leq \mathrm{wt}([L'])$. It is a consequence that

$$\mathrm{Ext}^i(L, L') = 0, \qquad \text{if } \mathrm{wt}([L]) - i < \mathrm{wt}([L']), \tag{2.2}$$

or, equivalently,

$$\mathrm{Ext}^i(L, L') \neq 0 \quad \Rightarrow \quad \mathrm{wt}([L]) \geq \mathrm{wt}([L']) + i. \tag{2.3}$$

In the following we mostly write $\mathrm{wt}(L)$ for $\mathrm{wt}([L])$. A *Tate twist* on a mixed category $\mathcal{A}$ is an autoequivalence $(1) : \mathcal{A} \to \mathcal{A}$ such that

$$\mathrm{wt}(L(1)) = \mathrm{wt}(L) - 1 \qquad \text{if } L \text{ is simple.}$$

Henceforth, all abelian categories will be mixed and equipped with a Tate twist. For more details on mixed categories with Tate twist we refer to [2, 4, 14].

2.2. **Weight filtration.** Recall the following standard fact ([2, Lemma 4.1.2]):

Lemma 2.1. *Let $\mathcal{A}$ be a mixed abelian category with Tate twist. Then any object $M \in \mathcal{A}$ has a unique finite filtration $0 = W_0 \subset W_1 \subset \ldots \subset W_r = M$ such that W_i/W_{i-1} is a direct sum of simple objects, all of weight i.*

This filtration is called the *weight filtration*. In case $W_i/W_{i-1} \neq 0$ we say i *occurs as weight* in M. If only one weight occurs, then M is called *pure* of this weight. In general, the maximal weight occurring in M is called the *degree* of M.

Note that morphisms are compatible with weight filtrations, in the sense that $f(W_i(M)) \subset W_i(N)$ for any morphism $f : M \to N$ between objects M, N in $\mathcal{A}$ with weight filtrations $W_\bullet(M)$ and $W_\bullet(N)$.

Example 2.2. Our standard example of a mixed abelian category is the category A-gmod of finite-dimensional graded modules over a finite-dimensional positively graded algebra $A = \oplus_{i \in \mathbb{Z}_{\geq 0}} A_i$ over the complex numbers with semisimple A_0. Each simple module L is concentrated in a single degree $-\mathrm{wt}(L)$ and the Tate twist (1) is given by shifting the degree up by 1. Any mixed abelian category can be realized as the category of modules over a projective limit of such positively graded algebras; see [2, 4.1.6] for a precise statement.

For $n \in \mathbb{Z}$, let $\mathcal{A}_{\leq n}$ (resp. $\mathcal{A}_n$, $\mathcal{A}_{\geq n}$), be the Serre subcategory of $\mathcal{A}$ generated by the simple objects of weight $\leq n$ (resp. n, $\geq n$). If $m \leq n$, we also put $\mathcal{A}_{[m,n]} = \mathcal{A}_{\geq m} \cap \mathcal{A}_{\leq n}$. For any $X \in \mathcal{A}$, the weight filtration (Lemma 2.1) defines a functorial short exact sequence

$$0 \to \beta_{\leq n} X \to X \to \beta_{\geq n+1} X \to 0$$

where $\beta_{\leq n} X$ has weights $\leq n$ and $\beta_{\geq n+1} X$ has weights $\geq n+1$. Moreover, the functors $\beta_{\leq n} : \mathcal{A} \to \mathcal{A}_{\leq n}$ and $\beta_{\geq n+1} : \mathcal{A} \to \mathcal{A}_{\geq n+1}$ are exact, so we can apply them to a chain complex $C^\bullet$ in $\mathcal{A}$ and get a short exact sequence of chain complexes. These functors induce derived functors $D^-(\mathcal{A}) \to D^-(\mathcal{A})$, so for any object $X \in D^-(\mathcal{A})$, there is a functorial distinguished triangle

$$\beta_{\leq n} X \to X \to \beta_{\geq n+1} X \to$$

in $D^-(\mathcal{A})$. (The same remarks apply to the bounded $D^{\mathrm{b}}(\mathcal{A})$ and bounded below $D^+(\mathcal{A})$ derived categories as well, but we will work primarily with $D^-(\mathcal{A})$.) These functors endow $D^-(\mathcal{A})$ with a *baric structure* in the sense of [1].

Definition 2.3. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between two mixed abelian categories. The *weight amplitude* of F is defined to be the infimum of the set

$$\{a \in \mathbb{Z} \mid a \geq 0 \text{ and } F(\mathcal{A}_{\leq n}) \subset \mathcal{B}_{\leq n+a} \text{ for all } n \in \mathbb{Z}\} \cup \{+\infty\}.$$

2.3. **Coefficients rings and Grothendieck groups.** Recall that the Grothendieck group of a triangulated category is defined as $K(\mathcal{C}) = F(\mathcal{C})/R(\mathcal{C})$, where $F(\mathcal{C})$ is the free abelian group on isomorphism classes $[M]$ of objects $M \in \mathcal{C}$, and $R(\mathcal{C})$ is the ideal generated by the expressions $[A] - [C] + [B]$ whenever there is a distinguished triangle of the form $A \to C \to B \to A[1]$. If $\mathcal{C}$ admits a bounded t-structure whose heart is $\mathcal{A}$, it can be shown that $K(\mathcal{A}) \cong K(\mathcal{C})$. (In the sequel, we work mostly with categories whose natural t-structure is *not* bounded, however.) Most Grothendieck groups we consider will naturally be modules over one of the following rings (two of which were mentioned in Section 1):

$$\mathfrak{R}_0 = \mathbb{Z}[q], \qquad \mathfrak{R} = \mathbb{Z}[q, q^{-1}], \qquad \hat{\mathfrak{R}}_0 = \mathbb{Z}[[q]], \qquad \hat{\mathfrak{R}} = \mathfrak{R} \otimes_{\mathfrak{R}_0} \hat{\mathfrak{R}}_0 = \mathbb{Z}[[q]][q^{-1}].$$

For instance, the Tate twist induces an automorphism $q : K(\mathcal{A}) \to K(\mathcal{A})$, where $[X(1)] = q[X]$, and so makes $K(\mathcal{A})$ into an $\mathfrak{R}$-module. It also restricts to a fully faithful, exact functor $(1) : \mathcal{A}_{\leq n} \to \mathcal{A}_{\leq n}$, but this is no longer an equivalence. The Grothendieck group $K(\mathcal{A}_{\leq n})$ is naturally an $\mathfrak{R}_0$-submodule of $K(\mathcal{A})$. It follows from (2.1) that $K(\mathcal{A})$ is free as an $\mathfrak{R}$-module (see [2, Lemma 4.3.2]).

In fact, for any $n \in \mathbb{Z}$, we have

$$K(\mathcal{A}_{\leq n}) \cong \mathfrak{R}_0[\mathrm{Irr}(\mathcal{A}_n)] \qquad \text{and} \qquad K(\mathcal{A}) \cong \mathfrak{R} \otimes_{\mathfrak{R}_0} K(\mathcal{A}_{\leq n}). \tag{2.4}$$

The $\mathfrak{R}_0$-module $K(\mathcal{A}_{\leq n})$ is equipped with a natural (q)-adic topology, in which the submodules

$$q^i \cdot K(\mathcal{A}_{\leq n}) = K(\mathcal{A}_{\leq n-i})$$

constitute a basis of neighborhoods around 0. Similarly, we endow $K(\mathcal{A})$ with a topology (also called "(q)-adic") by declaring the submodules $K(\mathcal{A}_{\leq i}) \subset K(\mathcal{A})$ to be a basis of neighborhoods around 0. Let

$$\hat{K}(\mathcal{A}_{\leq n}) \qquad \text{and} \qquad \hat{K}(\mathcal{A})$$

denote the completions of each of these modules in the (q)-adic topology. These completions are modules over $\hat{\mathfrak{R}}_0$ and $\hat{\mathfrak{R}}$, respectively.

2.4. **Definition of $D^\nabla(\mathcal{A})$.** Given a finite-length mixed abelian category with a Tate twist, we define the following full subcategory of $D^-(\mathcal{A})$:

$$D^\nabla(\mathcal{A}) = \left\{ X \in D^-(\mathcal{A}) \,\middle|\, \begin{array}{c} \text{for each } m \in \mathbb{Z}, \text{ only finitely many of the } H^i(X) \\ \text{contain a composition factor of weight} > m \end{array} \right\}.$$

It is easy to see that $D^\nabla(\mathcal{A})$ is closed under suspensions (or shifts) $[i]$, $i \in \mathbb{Z}$, and cones, and hence that it is a full triangulated subcategory of $D^-(\mathcal{A})$.

For $n \in \mathbb{Z}$, we also define the following full subcategories of $D^\nabla(\mathcal{A})$:

$$D^\nabla_{\leq n}(\mathcal{A}) = \{X \in D^\nabla(\mathcal{A}) \mid \text{for all } i \in \mathbb{Z},\ H^i(X) \text{ has weights} \leq n\},$$
$$D^\nabla_{\geq n}(\mathcal{A}) = \{X \in D^\nabla(\mathcal{A}) \mid \text{for all } i \in \mathbb{Z},\ H^i(X) \text{ has weights} \geq n\}.$$

If $m \leq n$, we also put $D^{\triangledown}_{[m,n]}(\mathcal{A}) = D^{\triangledown}_{\geq m}(\mathcal{A}) \cap D^{\triangledown}_{\leq n}(\mathcal{A})$. It follows from the definition of $D^{\triangledown}(\mathcal{A})$ that any object in $D^{\triangledown}_{\geq n}(\mathcal{A})$ has only finitely many nonzero cohomology objects, so

$$D^{\triangledown}_{\geq n}(\mathcal{A}) \subset D^{\mathrm{b}}(\mathcal{A}). \tag{2.5}$$

The Tate twist induces an autoequivalence $(1) : D^{\triangledown}(\mathcal{A}) \to D^{\triangledown}(\mathcal{A})$ and a fully faithful functor $(1) : D^{\triangledown}_{\leq n}(\mathcal{A}) \to D^{\triangledown}_{\leq n}(\mathcal{A})$, so $K(D^{\triangledown}(\mathcal{A}))$ and $K(D^{\triangledown}_{\leq n}(\mathcal{A}))$ are modules over $\mathfrak{R}$ and $\mathfrak{R}_0$, respectively. The categories $D^{\triangledown}_{\geq m}(\mathcal{A})$ are not preserved by the Tate twist, but nevertheless we will construct in the next section an $\mathfrak{R}_0$-module structure on its Grothendieck group.

3. The Grothendieck Group of $D^{\triangledown}(\mathcal{A})$

The main goal of this section is to prove part (1) of Theorem 1.1.

3.1. **Sequences of $\mathfrak{R}_0$-modules.** The categories $\mathcal{A}_{\geq m}$ and $D^{\triangledown}_{\geq m}(\mathcal{A})$ are not preserved by the Tate twist (1), so we use a different functor to make $K(\mathcal{A}_{\geq m})$ and $K(D^{\triangledown}_{\geq m}(\mathcal{A}))$ into $\mathfrak{R}_0$-modules: we put

$$q \cdot [X] = [\beta_{\geq m}(X(1))] \qquad \text{for } X \in \mathcal{A}_{\geq m} \text{ or } X \in D^{\triangledown}_{\geq m}(\mathcal{A}).$$

This definition also makes sense for $\mathcal{A}_{[m,n]}$ and $D^{\triangledown}_{[m,n]}(\mathcal{A})$.

Lemma 3.1. *For any $n \in \mathbb{Z}$, there is commutative diagram of $\mathfrak{R}_0$-modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(\mathcal{A}_{\leq n}) & \longrightarrow & K(\mathcal{A}) & \longrightarrow & K(\mathcal{A}_{\geq n+1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K(D^{\triangledown}_{\leq n}(\mathcal{A})) & \longrightarrow & K(D^{\triangledown}(\mathcal{A})) & \longrightarrow & K(D^{\triangledown}_{\geq n+1}(\mathcal{A})) & \longrightarrow & 0 \end{array} \tag{3.1}$$

in which the rows are short exact sequences. Moreover, the first two vertical maps are injective, and the last is an isomorphism.

Proof. We begin by treating the second row in this diagram. Consider the surjective map $\gamma : F(D^{\triangledown}(\mathcal{A})) \to F(D^{\triangledown}_{\leq n}(\mathcal{A})) \oplus F(D^{\triangledown}_{\geq n+1}(\mathcal{A}))$ defined as $[X] \mapsto ([\beta_{\leq n}X], [\beta_{\geq n+1}X])$. Given a distinguished triangle $A \to X \to B \to$ in $D^{\triangledown}(\mathcal{A})$, we have

$$\gamma([X]-[A]-[B]) = ([\beta_{\leq n}X]-[\beta_{\leq n}A]-[\beta_{\leq n}B], [\beta_{\geq n+1}X]-[\beta_{\geq n+1}A]-[\beta_{\geq n+1}B]).$$

Because the functors $\beta_{\leq n}$ and $\beta_{\geq n+1}$ are functors of triangulated categories, this calculation shows that $\gamma(R(D^{\triangledown}(\mathcal{A}))) \subset R(D^{\triangledown}_{\leq n}(\mathcal{A})) \oplus R(D^{\triangledown}_{\geq n+1}(\mathcal{A}))$. Since the restriction of $\beta_{\leq n}$, resp. $\beta_{\geq n+1}$, to $D^{\triangledown}_{\leq n}(\mathcal{A})$, resp. $D^{\triangledown}_{\geq n+1}(\mathcal{A})$, is the identity functor, we actually have $\gamma(R(D^{\triangledown}(\mathcal{A}))) = R(D^{\triangledown}_{\leq n}(\mathcal{A})) \oplus R(D^{\triangledown}_{\geq n+1}(\mathcal{A}))$. We conclude that γ induces an isomorphism of abelian groups

$$K(D^{\triangledown}(\mathcal{A})) = K(D^{\triangledown}_{\leq n}(\mathcal{A})) \oplus K(D^{\triangledown}_{\geq n+1}(\mathcal{A})). \tag{3.2}$$

In this direct sum, the inclusion $K(D^{\triangledown}_{\leq n}(\mathcal{A})) \to K(D^{\triangledown}(\mathcal{A}))$ is in fact induced by the inclusion functor $\iota : D^{\triangledown}_{\leq n}(\mathcal{A}) \to D^{\triangledown}(\mathcal{A})$, since $\beta_{\leq n} \circ \iota = \mathrm{id}$ and $\beta_{\geq n+1} \circ \iota = 0$. Since ι commutes with the Tate twist, the inclusion map $K(D^{\triangledown}_{\leq n}(\mathcal{A})) \to K(D^{\triangledown}(\mathcal{A}))$ is a homomorphism of $\mathfrak{R}_0$-modules. On the other hand, the setup is such that $[\beta_{\geq n+1}] : K(D^{\triangledown}(\mathcal{A})) \to K(D^{\triangledown}_{\geq n+1}(\mathcal{A}))$ commutes with the action of $q \in \mathfrak{R}_0$ on both groups. Thus, (3.2) gives rise to the desired short exact sequence (the second line of (3.1)) of $\mathfrak{R}_0$-modules.

Since $\beta_{\leq n}$ and $\beta_{\geq n+1}$ are t-exact, the same argument can be repeated with the abelian categories $\mathcal{A}_{\leq n}$, $\mathcal{A}$, and $\mathcal{A}_{\geq n+1}$, yielding the $\hat{\mathfrak{R}}_0$-module structure and exactness of the first row in the diagram (3.1). Since all maps in that diagram are induced by inclusion functors or by $\beta_{\geq n+1}$, it is easy to see that the diagram commutes.

Recall from (2.5) that $D^{\nabla}_{\geq n+1}(\mathcal{A}) \subset D^{\mathrm{b}}(\mathcal{A})$. The fact that $K(\mathcal{A}_{\geq n+1}) \to K(D^{\nabla}_{\geq n+1}(\mathcal{A}))$ is an isomorphism follows from the fact that $\mathcal{A}_{\geq n+1}$ is the heart of a *bounded* t-structure on $D^{\nabla}_{\geq n+1}(\mathcal{A})$.

Next, consider an element $f \in K(\mathcal{A})$. We may write $f = a_1[X_1] + \cdots + a_k[X_k]$ for suitable simple objects $X_i \in \mathcal{A}$. Now, choose n such that $n < \mathrm{wt}(X_i)$ for all i. It is clear from Lemma 2.1 that $[\beta_{\geq n+1}]f \neq 0$. In view of the preceding paragraph, it follows from the commutativity of (3.1) that the image of f in $K(D^{\nabla}(\mathcal{A}))$ is nonzero. Thus, the middle vertical arrow in (3.1) is injective. The injectivity of the first vertical arrow is then clear as well. □

The same reasoning yields the following related statement.

Lemma 3.2. *Supose $m \leq n$. There is a commutative diagram of $\mathfrak{R}_0$-modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(\mathcal{A}_{\leq m}) & \longrightarrow & K(\mathcal{A}_{\leq n}) & \longrightarrow & K(\mathcal{A}_{[m+1,n]}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K(D^{\nabla}_{\leq m}(\mathcal{A})) & \longrightarrow & K(D^{\nabla}_{\leq n}(\mathcal{A})) & \longrightarrow & K(D^{\nabla}_{[m+1,n]}(\mathcal{A})) & \longrightarrow & 0 \end{array} \tag{3.3}$$

in which the rows are short exact sequences. Moreover, the first two vertical maps are injective, and the last is an isomorphism. □

3.2. **(q)-adic topology.** Recall from (3.1) that $K(D^{\nabla}_{\leq m}(\mathcal{A}))$ can naturally be identified with a submodule of $K(D^{\nabla}(\mathcal{A}))$ (or of $K(D^{\nabla}_{\leq n})$, if $m \leq n$). Thus, we are at last able to define the (q)-adic topology on these modules: we take the set of submodules of the form $K(D^{\nabla}_{\leq m})$ to be a basis of neighborhoods of 0. It follows from the proof of Lemma 3.1 that for $s \in \mathbb{Z}$

$$q^s \cdot K(D^{\nabla}_{\leq m}(\mathcal{A})) = K(D^{\nabla}_{\leq m-s}(\mathcal{A})), \tag{3.4}$$

so $K(D^{\nabla}_{\leq n}(\mathcal{A}))$ and $K(D^{\nabla}(\mathcal{A}))$ are naturally topological $\mathfrak{R}_0$- and $\mathfrak{R}$-modules, respectively.

Lemma 3.3. *The $\mathfrak{R}_0$-module $K(D^{\nabla}_{\leq n}(\mathcal{A}))$ is complete in the (q)-adic topology. Indeed, the natural map $K(\mathcal{A}_{\leq n}) \to K(D^{\nabla}_{\leq n}(\mathcal{A}))$ induces an isomorphism*

$$\hat{K}(\mathcal{A}_{\leq n}) \to K(D^{\nabla}_{\leq n}(\mathcal{A})).$$

Proof. Since $q^s \cdot K(D^{\nabla}_{\leq n}(\mathcal{A})) = K(D^{\nabla}_{\leq n-s}(\mathcal{A}))$ for any $s \in \mathbb{Z}_{\geq 0}$, it follows from Lemma 3.2, with $m = n - s$, that

$$K(D^{\nabla}_{\leq n}(\mathcal{A}))/q^s \cdot K(D^{\nabla}_{\leq n}(\mathcal{A})) \cong K(D^{\nabla}_{[n-s+1,n]}(\mathcal{A})).$$

Suppose we have a sequence of elements $f_i \in K(D^{\nabla}_{[n-i+1,n]}(\mathcal{A}))$, $i \in \mathbb{Z}_{\geq 0}$ satisfying the condition that $[\beta_{\geq n-j+1}]f_i = f_j$ when $j < i$. To show that $K(D^{\nabla}_{\leq n}(\mathcal{A}))$ is complete, we must exhibit an element g such that $[\beta_{\geq n-i+1}]g = f_i$ for all i.

By Lemma 3.2 again, we identify $K(D^{\nabla}_{[n-i+1,n]}(\mathcal{A}))$ with $K(\mathcal{A}_{[n-i+1,n]})$, viewed as a subgroup of $K(\mathcal{A})$. Regarding all the f_i as elements of $K(\mathcal{A})$, we can form the elements

$$a_i = f_i - f_{i-1} = f_i - [\beta_{\geq n-i+2}]f_i \in K(\mathcal{A}_{n-i+1}).$$

Then $f_i = a_1 + a_2 + \cdots + a_i$ for all i. Since $K(\mathcal{A}_{n-i+1})$ is the free abelian group on $\mathrm{Irr}(A_{n-i+1})$, we can write

$$a_i = c_{i1}[L_{i1}] + \cdots + c_{i,r_i}[L_{i,r_i}] - d_{i1}[M_{i1}] - \cdots - d_{i,s_i}[M_{i,s_i}]$$

for unique (up to renumbering) $[L_{ij}], [M_{ij}] \in \mathrm{Irr}(\mathcal{A}_{n-i+1})$, and $c_{ij}, d_{ij} > 0$. Now, let $X^\bullet$ be the chain complex with trivial differentials and

$$X^k = \begin{cases} 0 & \text{if } k \geq 0, \\ \bigoplus_{j=1}^{r_i} L_{ij}^{\oplus c_{ij}} & \text{if } k = -2i < 0 \text{ is even}, \\ \bigoplus_{j=1}^{s_i} M_{ij}^{\oplus d_{ij}} & \text{if } k = -2i+1 < 0 \text{ is odd}. \end{cases} \tag{3.5}$$

By construction, $H^k(X^\bullet) \cong X^k$ vanishes for $k \geq 0$, and is pure of weight $n + 1 + \lfloor k/2 \rfloor$ for $k < 0$, so $X^\bullet \in D^{\nabla}_{\leq n}(\mathcal{A})$. It is easy to see that $[\beta_{\geq n-i+2}X^\bullet] = f_i$, so $g = [X^\bullet]$ is the element we were looking for.

Finally, we also see from Lemma 3.2 that

$$K(\mathcal{A}_{\leq n})/q^i \cdot K(\mathcal{A}_{\leq n}) \cong K(D^{\nabla}_{\leq n}(\mathcal{A}))/q^i \cdot K(D^{\nabla}_{\leq n}(\mathcal{A}))$$

for each i, so $K(\mathcal{A}_{\leq n})$ and $K(D^{\nabla}_{\leq n}(\mathcal{A}))$ have the same completion. □

Proof of Theorem 1.1(1). The injectivity of $K(\mathcal{A}) \to K(D^{\nabla}(\mathcal{A}))$ was established in Lemma 3.1. Since every Cauchy sequence in $K(\mathcal{A})$ or $K(D^{\nabla}(\mathcal{A}))$ is contained in some submodule $K(\mathcal{A}_{\leq n})$ or $K(D^{\nabla}_{\leq n}(\mathcal{A}))$, Lemma 3.3 implies that $K(D^{\nabla}(\mathcal{A}))$ is complete, and that $\hat{K}(\mathcal{A}) \to K(D^{\nabla}(\mathcal{A}))$ is an isomorphism. □

4. Projective Resolutions and Derived Functors

We will prove the remaining parts of Theorem 1.1 in this section. Henceforth, $\mathcal{A}$ is assumed to have enough projectives. Since $\mathcal{A}$ is also assumed to be a finite-length category, Fitting's lemma and its consequences hold; for instance, each projective is a direct sum of finitely many indecomposable ones. The degree of an indecomposable projective P can also be defined as the integer

$$\deg(P) = \mathrm{wt}(\text{the unique simple quotient of } P).$$

For $n \in \mathbb{Z}$, let $\mathrm{Proj}(\mathcal{A})_n$ denote the set of isomorphism classes of indecomposable projectives of degree n. Obviously, the map $P \mapsto P/\operatorname{rad} P$ induces a bijection

$$\mathrm{Proj}(\mathcal{A})_n \xrightarrow{\sim} \mathrm{Irr}(\mathcal{A}_n).$$

By considering the weight filtration (Lemma 2.1, see also Example 2.2), one can see that

$$\deg(P) = n \qquad \text{implies} \qquad P \in \mathcal{A}_{\leq n}. \tag{4.1}$$

More generally, the degree of a projective object is simply the maximum of the degrees of its indecomposable summands, and the degree of an arbitrary object is the degree of its projective cover.

Definition 4.1. A bounded-above complex $P^\bullet$ of projectives is said to have *asymptotically decreasing weights* if for each $m \in \mathbb{Z}$, all but finitely many of the terms P^i have degree $\leq m$.

Lemma 4.2. *The following conditions on an object $X \in D^-(\mathcal{A})$ are equivalent:*

(1) $X \in D^{\nabla}_{\leq n}(\mathcal{A})$.
(2) *X is quasi-isomorphic to a bounded-above complex of projectives $P^\bullet$ with asymptotically decreasing weights where each term P^i is of degree $\leq n$.*

Proof. In view of (4.1), it is obvious that condition (2) implies condition (1). For the other implication, we first consider the special case where $X \in \mathcal{A}$. Let d be the degree of X, and let $Q^\bullet$ be a minimal projective resolution of X. The projective cover of a simple object L occurs as a direct summand of P^i (for $i \leq 0$) if and only if $\mathrm{Ext}^{-i}(X, L) \neq 0$. This can only happen if $\mathrm{wt}(L) \leq d + i$, so Q^i is of degree $\leq d + i$. Using (4.1) again, we see that the complex $Q^\bullet$ satisfies condition (2).

For general X, choose a minimal projective resolution $Q^\bullet_i$ for each cohomology object $H^i(X)$. Then X is quasi-isomorphic to a complex $P^\bullet$ with terms of the form

$$P^i = \bigoplus_{k \geq 0} Q^{-k}_{i+k}.$$

Let N be the largest integer such that $H^N(X) \neq 0$. (Such an N exists because X is bounded above.) For $i \leq N$, let d_i be the largest weight that occurs in $H^i(X)$, or let $d_i = -\infty$ if $H^i(X) = 0$. Next, let

$$a_i = \max\{d_i, d_{i+1} - 1, d_{i+2} - 2, \ldots, d_N - (N - i)\}.$$

Note that P^i is of degree $\leq a_i$. In particular, each P^i is of degree $\leq n$. Next, given $m \in \mathbb{Z}$, there is a k_0 such that $d_i \leq m$ for all $i \leq k_0$. Let

$$k = \min\{i - d_i + m \mid k_0 \leq i \leq N\}.$$

We claim that for all $i \leq k$, $a_i \leq m$. Indeed, if $i \leq k$ and $0 \leq j \leq N - i$, then

$$\begin{aligned} &d_{i+j} - j \leq d_{i+j} \leq m && \text{if } i + j \leq k_0, \\ &d_{i+j} - j = i + m - (i + j - d_{i+j} + m) \leq i + m - k \leq m && \text{if } k_0 \leq i + j \leq N. \end{aligned}$$

Hence $P^\bullet$ is a bounded above complex of projectives with asymptotically decreasing weights, so it satisfies condition (2), as desired. □

Lemma 4.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact functor commuting with Tate twist. If F has weight amplitude $\alpha < \infty$, then the left-derived functor $\mathcal{L}F : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ has the property that*

$$\mathcal{L}F(D^{\nabla}_{\leq n}(\mathcal{A})) \subset D^{\nabla}_{\leq n+\alpha}(\mathcal{B}). \tag{4.2}$$

Proof. Given $X \in D^{\nabla}_{\leq n}(\mathcal{A})$, choose a projective resolution $P^\bullet$ satisfying the condition in Lemma 4.2(2). It is clear from (4.1) that all terms of the complex $F(P^\bullet)$ have weights $\leq n + \alpha$, and that for any m, only finitely terms have a composition factor of weight $> m$. The same then holds for its cohomology objects, so $\mathcal{L}F(X) \in D^{\nabla}_{\leq n+\alpha}(\mathcal{B})$. □

Proof of Theorem 1.1(2). Given $m \in \mathbb{Z}$, let k be such that for all $i \leq k$, the cohomology $H^i(X)$ has weights $\leq m$. Then $[X] = [\tau^{\leq k} X] + [\tau^{\geq k+1} X]$ where $\tau^{\leq k}$, $\tau^{\geq k}$ denote the usual truncation functors in triangulated categories. Since X is bounded above, $\tau^{\geq k+1} X$ has only finitely many nonzero cohomology objects,

and it is clear that $[\tau_{\geq k+1}X] = \sum_{i=k+1}^{\infty}(-1)^i[H^i(X)]$. Moreover, by construction, $\tau^{\leq k}X \in D^{\nabla}_{\leq m}(\mathcal{A})$, so

$$[X] - \sum_{i=k+1}^{\infty}(-1)^i[H^i(X)] \in K(\mathcal{A}_{\leq m}).$$

Thus, the series $\sum(-1)^i[H^i(X)]$ converges to $[X]$. The argument for $\sum(-1)^i[P^i]$ is similar. □

Proof of Theorem 1.1(3). A description of the completion of a free module can be found in [15, §2.4]. It follows from that description that the basis of a free $\mathfrak{R}_0$-module spans a dense free $\hat{\mathfrak{R}}_0$-submodule of its completion, and that the two coincide if the basis is finite. Thus, it follows from (2.4) that $\mathrm{Irr}(\mathcal{A}_0)$ spans a dense free $\hat{\mathfrak{R}}_0$-submodule of $\hat{K}(\mathcal{A}_{\leq 0})$.

The case of $\mathrm{Proj}(\mathcal{A})_0$ is somewhat different, since this set does *not* give an $\mathfrak{R}_0$-basis for $K(\mathcal{A}_{\leq 0})$ in general. However, recall that if $P \in \mathrm{Proj}(\mathcal{A})_0$ and if $L \in \mathrm{Irr}(\mathcal{A}_0)$ is its unique irreducible quotient, then in $K(\mathcal{A}_{\leq 0})$, we have

$$[P] = [L] + (\text{terms in } q \cdot K(\mathcal{A}_{\leq 0})).$$

It is easy to deduce from this that the elements of $\mathrm{Proj}(\mathcal{A})_0$ are linearly independent in $K(\mathcal{A}_{\leq 0})$: any relation would give rise to a relation among elements of $\mathrm{Irr}(\mathcal{A}_0)$. Thus, the $\mathfrak{R}_0$-submodule $K_{\mathrm{pf}}(\mathcal{A}_{\leq 0}) \subset K(\mathcal{A}_{\leq 0})$ generated by $\mathrm{Proj}(\mathcal{A})_0$ is free. It follows that the corresponding $\mathfrak{R}$-submodule $K_{\mathrm{pf}}(\mathcal{A}) \subset K(\mathcal{A})$ is free as well. Since completion is left-exact, we have a natural inclusion $\hat{K}_{\mathrm{pf}}(\mathcal{A}) \subset \hat{K}(\mathcal{A})$. The argument of the previous paragraph shows that $\mathrm{Proj}(\mathcal{A})_0$ spans a free dense submodule of $\hat{K}_{\mathrm{pf}}(\mathcal{A})$, so it remains only to show that this submodule is also dense in $\hat{K}(\mathcal{A})$. But this follows from the fact that the class of every object in $D^{\nabla}(\mathcal{A})$ can be written as a convergent series of projectives. □

Proof of Theorem 1.1(4). We see from Lemma 4.3 that $\mathcal{L}F(D^{\nabla}(\mathcal{A})) \subset D^{\nabla}(\mathcal{B})$, so we certainly have an induced map $[\mathcal{L}F] : K(D^{\nabla}(\mathcal{A})) \to K(D^{\nabla}(\mathcal{B}))$. The assertion that it is continuous is then just a restatement of (4.2). □

5. Examples and Applications

5.1. **Graded modules over a graded local ring.** Let k be a field and $\mathbb{H} = \bigoplus_{i\in\mathbb{Z}_{\geq 0}}\mathbb{H}^i$ a finite-dimensional positively graded connected (i.e., $\mathbb{H}^0 = k$) k-algebra. Then $\mathbb{H}$ is graded local with maximal ideal $\mathfrak{m} = \bigoplus_{i\in\mathbb{Z}_{>0}}\mathbb{H}^i$ and has, up isomorphism and grading shift, a unique irreducible (finite-dimensional) graded $\mathbb{H}$-module, namely the trivial module $L = k = \mathbb{H}^0$. Let $\mathbb{H}$-gmod be the category of finite dimensional $\mathbb{Z}$-graded $\mathbb{H}$-modules with grading shift functor $\langle j\rangle$ defined as $(\langle j\rangle M)^i = M^{i-j}$ for $M = \bigoplus_{i\in\mathbb{Z}} M^i \in \mathbb{H}$-gmod.

Proposition 5.1.
- $\mathbb{H}$-gmod *with* $\mathrm{wt}(\langle i\rangle L) = -i$ *and* $(1) = \langle 1\rangle$ *is a noetherian and artinian mixed abelian category with Tate twist.*
- *Let* $p(q) = \sum_{i\geq 0}(\dim \mathbb{H}^i)q^i$ *be the Poincaré polynomial of* $\mathbb{H}$. *It has nontrivial constant term, so it can be inverted in the ring* $\hat{\mathfrak{R}}$. *In fact,* $[L] = p(q)^{-1}[\mathbb{H}]$ *in* $\hat{K}(\mathcal{A})$, *and each of* $[L]$ *and* $[\mathbb{H}]$ *gives an* $\hat{\mathfrak{R}}$*-basis for* $\hat{K}(\mathcal{A})$.

A natural example arising in this context is the cohomology ring $\mathbb{H} = H^*(X)$ of a smooth projective complex algebraic variety X. If we choose for instance $X = \mathbb{CP}^1$

then $\mathbb{H} = H^*(X) = \mathbb{C}[x]/(x^2)$ with Poincare polynomial $p(q) = 1 + q^2$, and we obtain the equation $[L] = \frac{1}{1+q^2}[\mathbb{H}] = (1 - q^2 + q^4 - q^6 + \ldots)[\mathbb{H}]$ in $\hat{K}(\mathcal{A})$.

More generally, if $\mathbb{H} = H^*(X)$, where $X = \mathrm{Gr}(i, n)$ is the Grassmannian variety of complex i-planes in $\mathbb{C}^n$, or any partial flag variety $X = \mathrm{GL}(n, \mathbb{C})/P$ for some parabolic subgroup P, then the complex cohomology rings $H^*(X)$ are explicitly known (see for instance [7], [8]). We have the equality $[\mathbb{H}] = \binom{n}{d_1,\ldots,d_r}[\mathbb{C}]$ in the Grothendieck group of graded $\mathbb{H}$-modules, where

$$\binom{n}{d_1, \ldots, d_r} = \frac{[n]!}{[d_1]![d_2]! \cdots [d_r]![(n - d_1 - \cdots - d_r)]!}$$

denotes the *quantum binomial coefficient* defined by taking the quantum numbers $[n] = \frac{q^{2n}-1}{q^2-1} = 1 + q^2 + \cdots + q^{2(n-1)}$ for $n \in \mathbb{Z}_{>0}$ and their factorials $[n]! = [1][2][3] \cdots [n]$ with $[0]! = 1$. Interpreting this quantum binomial coefficient as a formal power series in q, we obtain the equation

$$[L] = \frac{1}{\binom{n}{d_1,\ldots,d_r}}[\mathbb{H}]$$

in $\hat{K}(\mathcal{A})$. By Theorem 1.1, L and $[\mathbb{H}]$ each form an $\hat{\mathfrak{R}}$-basis of $\hat{K}(\mathcal{A})$, and the transformation matrix is given by quantum binomial coefficients and their inverses. This transformation matrix also occurs in the representation theory of the smallest quantum group $U_q(\mathfrak{sl}_2)$, as we will see in the next section.

5.2. **Categorification of finite-dimensional irreducible modules for quantum $\mathfrak{sl}_2$.** Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q. Let $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ be the associative algebra over $\mathbb{C}(q)$ generated by E, F, K, K^{-1} subject to the relations:

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK,$$
$$EF - FE = \tfrac{K-K^{-1}}{q-q^{-1}}.$$

Let $\bar{V}_n$ be the unique (up to isomorphism) irreducible module for $\mathfrak{sl}_2$ of dimension $n+1$. Denote by V_n its quantum analogue (of type I), that is the irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module with basis $\{v_0, v_1, \ldots, v_n\}$ such that

$$K^{\pm 1}v_i = q^{\pm(2i-n)}v_i \qquad Ev_i = [i+1]q^{-i-1}v_{i+1} \qquad Fv_i = [n-i+1]q^{1-i}v_{i-1}. \tag{5.1}$$

Note that it is defined over $\mathfrak{R}$. The chosen basis is the canonical basis in Lusztig's theory of canonical bases ([11], [5]) and pairs via a bilinear form with the *dual canonical basis* given by $v^i = q^{-i(n-i)}\frac{1}{\binom{n}{i,n-i}}v_i$. Hence, passing to the completion $\hat{V}_n$ of V_n we have an isomorphism of $\hat{\mathfrak{R}}$-modules

$$\begin{aligned} \hat{V}_n &\mapsto \bigoplus_{i=0}^{n} \hat{K}(\mathcal{A}_i) \\ v_i &\mapsto [H^*(\mathrm{Gr}(i,n))\langle i(n-i)\rangle] \\ v^i &\mapsto [L_i] \end{aligned} \tag{5.2}$$

where $\mathcal{A}_i$ denotes the mixed abelian category $H^*(\mathrm{Gr}(i,n))$-gmod with unique simple object L_i of weight zero. The action of the quantum group can then be realized via correspondences: if we let $\mathrm{Gr}(i, i+1, n)$ be the variety of partial flags

$$(i\text{-plane}) \subset ((i+1)\text{-plane}) \subset \mathbb{C}^n,$$

then $H^*(\mathrm{Gr}(i,i+1,n))$ is naturally a $(H^*(\mathrm{Gr}(i,n)), H^*(\mathrm{Gr}(i+1,n))$-bimodule or a $(H^*(\mathrm{Gr}(i+1,n)), H^*(\mathrm{Gr}(i,n))$-bimodule. Tensoring (with appropriate grading shifts) with these bimodules defines exact endofunctors on $\oplus_{i=0}^n(\mathcal{A}_i)$ which induce the action of E and F on $\oplus_{i=0}^n\hat{K}(\mathcal{A}_i)$ given by the formula (5.1) via the isomorphism (5.2). For details see [6, Section 6] and [3] in the non-graded version.

5.3. **Quotient categories.** Let k be an algebraically closed field and $A = \bigoplus_{i\in\mathbb{Z}_{\geq 0}}$ a finite-dimensional positively graded k-algebra, semisimple in degree zero, i.e. $A_0 = k \oplus k \oplus \cdots \oplus k = k^{\oplus r}$. Let $A = \oplus_{i=1}^r Ae_i$ be the decomposition into indecomposable projective modules with simple quotients L_i, $1 \leq i \leq r$. Let $\mathcal{A} = A$-gmod be the mixed category of finite-dimensional graded A-modules with Tate twist $(1) = \langle -1\rangle$. Assume we are given a Serre subcategory $\mathscr{S}_I$ of $\mathcal{A}$ stable under Tate twist. That is, $\mathscr{S}_I$ is a full subcategory consisting of all modules which have composition factors only of the form $L_i\langle j\rangle$, where $j \in \mathbb{Z}$ and $i \in I$ for some fixed subset I of $\{1,\ldots,r\}$. Let

$$Q : \mathcal{A} \to \mathcal{A}/\mathscr{S}_I$$

be the quotient functor to the Serre quotient $\mathcal{A}/\mathscr{S}_I$. Under the identification of $\mathcal{A}/\mathscr{S}_I$ with graded modules over $\mathrm{End}_A(P_I)$, where $P_I = \bigoplus_{i\in I} Ae_i$ we have $Q = \mathrm{Hom}_A(\bigoplus_{i\in I} Ae_i, -)$. In particular, Q is exact and has left adjoint $Q' : M \mapsto M \otimes_{\mathrm{End}_A(P_I)} P_I$. Now the following is just a direct application of our main result:

Proposition 5.2. *The functors $Q : \mathcal{A} \to \mathcal{A}/\mathscr{S}_I$ and $Q' : \mathcal{A}/\mathscr{S}_i \to \mathcal{A}$ are exact and right exact respectively, commute with Tate twist and have finite weight amplitude. Hence the functors Q and $\mathcal{L}Q'$ induce continuous homomorphisms of $\hat{\mathfrak{R}}$-modules*

$$[Q] : \hat{K}(\mathcal{A}) \to \hat{K}(\mathcal{A}/\mathscr{S}_I), \quad [\mathcal{L}Q'] : \hat{K}(\mathcal{A}/\mathscr{S}_I) \to \hat{K}(\mathcal{A}).$$

Note that $(\mathcal{L}Q' \circ Q)^2 \cong \mathcal{L}Q' \circ Q$, since $Q \circ \mathcal{L}Q' \cong \mathrm{id}$. In [10] this property is used to categorify the Jones–Wenzl projectors $\hat{V}_i \otimes \hat{V}_j \to \hat{V}_k$ for any summand V_k of $V_i \otimes V_j$.

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