Algebraic Groups
Summer Semester 2008

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Exercise sheet 9

Solutions to be handed in on Wednesday 11th June 2008

Problem 32. Let H be a closed subgroup of a diagonalizable group G. Show that restriction of characters $\phi^* : X^*(G) \to X^*(H)$ is surjective.

Problem 33. (Rigidity of diagonalizable groups) Let G and H be diagonalizable groups and X a connected affine variety. Let $\phi : X \times G \to H$ be a morphism of varieties such that for any $x \in X$, the map $\phi_x : G \to H$, $g \mapsto \phi(x, g)$, defines a group homomorphism. Show that $\phi_x = \phi_y$ for all $x, y \in X$.

Hint: For any $\chi \in X^*(H)$, $\phi^*(\chi)$ can be written as a (finite) sum $\phi^*(\chi) = \sum_{\psi \in X^*(G)} f_{\chi,\psi} \otimes \psi$ where the $f_{\chi,\psi} \in k[X]$ are uniquely determined. Show and use that $f_{\chi,\psi} f_{\chi,\psi'} = \delta_{\psi,\psi'} f_{\chi,\psi}$.

Problem 34. Let Γ be a finitely generated abelian group.

(a) Show that $k\Gamma$ has nilpotent elements $\neq 0$ if and only if $\operatorname{char} k = p > 0$ and Γ has p-torsion.

Hint: reduce to the cases $\Gamma = \mathbb{Z}$ and $\Gamma = \mathbb{Z}/n\mathbb{Z}$.

Assume that Γ has no p-torsion if $\operatorname{char} k = p > 0$.

(b) Show that there is a diagonalizable affine algebraic group G with $X^*(G) \cong \Gamma$ by reduction to the cases $\Gamma = \mathbb{Z}$ and $\Gamma = \mathbb{Z}/n\mathbb{Z}$.

(c) Show the same result using the Hopf algebra structure on $k\Gamma$.

Problem 35. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between categories $\mathcal{A}$ and $\mathcal{B}$. Show: If F is fully faithful (bijective on morphism spaces) and essentially surjective (surjective on isomorphism classes of objects), then F is an equivalence.

(The converse is also true.)