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Exercise sheet 8

Solutions to be handed in on Wednesday 4th June 2008

Problem 28. Let G be an affine algebraic group. Show: If $G = (G, G)$ then $X^*(G) = 0$.

Problem 29. Let U be a unipotent affine algebraic group.

(a) Show that $X_*(U) = 0$ and $X^*(U) = 0$.
(b) Deduce that $X_*(\mathbb{G}_\mathrm{a}) = 0$ and $X^*(\mathbb{G}_\mathrm{a}) = 0$.

Problem 30. Let D_n be the closed subgroup of diagonal matrices in $\mathrm{GL}(n, k)$, and define $X_* = X_*(D_n)$, $X^* = X^*(D_n)$.

(a) Show that $X^* \cong \mathbb{Z}^n$ and $X_* \cong \mathbb{Z}^n$.
(b) Show: For $\chi \in X^*$ and $\alpha \in X_*$ there is a unique integer $n \in \mathbb{Z}$ such that $\chi(\alpha(z)) = z^n$ for all $z \in \mathbb{G}_\mathrm{m}$. We denote this integer by $\langle \chi, \alpha \rangle$.
(c) Show that

$$\begin{aligned} \langle \cdot, \cdot \rangle : X^* \times X_* &\to \mathbb{Z}, \\ (\chi, \alpha) &\mapsto \langle \chi, \alpha \rangle, \end{aligned}$$

is a perfect pairing between X^* and X_*, i. e. any group homomorphism $X^* \to \mathbb{Z}$ is of the form $\chi \mapsto \langle \chi, \alpha \rangle$ for a unique $\alpha \in X_*$, and similarly for X_*.