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Summer Semester 2008 | Olaf Schnürer

Exercise sheet 7

Solutions to be handed in on Wednesday 28th May 2008

Problem 23. Let V and W be finite dimensional vector spaces, $x \in \mathrm{GL}(V)$ and $y \in \mathrm{GL}(W)$. Prove that $(x \otimes y)_{\mathrm{s}} = x_{\mathrm{s}} \otimes y_{\mathrm{s}}$ and $(x \otimes y)_{\mathrm{u}} = x_{\mathrm{u}} \otimes y_{\mathrm{u}}$.

Problem 24. Show: There are fields F and $a \in F$ where the matrix $\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$ has no multiplicative Jordan decomposition.

Problem 25. Show that the set of semisimple elements of $\mathrm{GL}(2, k)$ is neither closed nor open. Is it a subgroup?

Problem 26. (Kostant-Rosenlicht) Let G be a unipotent affine algebraic group and X an affine G-variety. Prove that all G-orbits are closed in X.

Hint: Consider the action of G on $k[X]$ by left translations, $(g, f) \mapsto \lambda(g)(f)$, where $(\lambda(g)(f))(x) = f(g^{-1}x)$. Let Y be an orbit and $R = \overline{Y} - Y$. The ideal $\mathcal{I}(\overline{Y})$ is strictly smaller than $\mathcal{I}(R)$, and G acts on the quotient $\mathcal{I}(R)/\mathcal{I}(\overline{Y})$. Apply a theorem from the lecture and deduce that R must be empty.

Problem 27. Let $n \in \mathbb{N}$. A **partition** of n is a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of non-negative integers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ with $\sum \lambda_i = n$. If λ and μ are partitions of n, we write $\lambda \geq \mu$ if

$$\sum_{1 \leq i \leq l} \lambda_i \geq \sum_{1 \leq i \leq l} \mu_i$$

for all $1 \leq l \leq n$.

To a given nilpotent matrix $A \in \mathrm{M}(n \times n, k)$ we associate the partition $\lambda(A) = (\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ denotes the sizes of the Jordan blocks of A.

Let A, $B \in \mathrm{M}(n \times n, k)$ be nilpotent.

(a) Prove: $\lambda(A) \geq \lambda(B) \Leftrightarrow \mathrm{rank}(A^l) \geq \mathrm{rank}(B^l)$ for all $l \in \mathbb{N}$.

(b) Consider the operation of $G = \mathrm{GL}(n, k)$ on $\mathrm{M}(n \times n, k)$ by conjugation. We write $G.A$ for the G-orbit of A. Show: If $\overline{G.A} \subset \overline{G.B}$ then $\lambda(A) \leq \lambda(B)$. Hint: Consider the closed G-invariant subsets $Z_d = \{X \in \mathrm{M}(n \times n, k) \mid \mathrm{rank}\, X \leq d\}$.

(c) For $n = 3$ show: $\lambda(A) \leq \lambda(B)$ if and only if $\overline{G.A} \subset \overline{G.B}$.