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Exercise sheet 6

Solutions to be handed in on Wednesday 21th May 2008

Problem 19. Show that the open subvariety $k^2 - \{(0,0)\}$ of k^2 is not affine.

Problem 20. The natural action of $G = \mathrm{GL}(n,k)$ on k^n induces an action of G on $X = \mathbb{P}^{n-1}(k)$. Show that this action turns X into a G-variety.

Problem 21. Let $X = \mathbb{P}^2(k)$. Show:

(a) $\mathcal{O}_X(X) = k$.

(b) Any morphism $X \to Y$ into an affine variety Y is constant.

Problem 22. Let $V = k^n$. A flag in V is a sequence $F = (F_0, \dots, F_n)$ of subspaces

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V$$

with $\dim F_i = i$. Given $g \in G = \mathrm{GL}(n,k)$ and a flag F as above, we define $g.F := (g(F_0), g(F_1), \dots, g(F_n))$. This defines an action of G on the set $\mathcal{F}$ of flags in V. (In fact, $\mathcal{F}$ is a G-variety.)

(a) Let $T \subset \mathrm{GL}(n,k)$ be the subgroup of diagonal matrices. Determine the (finite) set of T-fixed points

$$\mathcal{F}^T = \{F \in \mathcal{F} \mid t.F = F \text{ for all } t \in T\}.$$

(b) Let $B \subset \mathrm{GL}(n,k)$ be the subgroup of upper triangular matrices. Determine the B-orbits in $\mathcal{F}$ and show that each B-orbit contains precisely one T-fixed point.

Hint: We say that a $(n \times n)$-matrix has echelon form if there is a permutation $\sigma \in S_n$ such that the i-th column has a 1 in the $\sigma(i)$-th row and below and to the right of this 1 only zeros (for $1 \leq i \leq n$). For example

$$\begin{bmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is in echelon form. Let E be the set of matrices in echelon form. Given $A \in E$, define $F_i(A)$ to be the span of the first i columns of A. Then $A \mapsto (F_0(A), F_1(A), \dots, F_n(A))$ defines a bijection $E \to \mathcal{F}$.