Algebraic Groups
Summer Semester 2008

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Exercise sheet 5

Solutions to be handed in on Wednesday 7th May 2008

Exercise 15. Let H and K be closed subgroups of an affine algebraic group G. We know: The commutator subgroup (H, K) is connected if H or K is connected. Is this statement still true if we drop the connectedness assumption?

Exercise 16.

(a) Show that $\mathrm{GL}(n, k)$ is connected.
Hint: Use that $\mathrm{Mat}(n \times n, k) \cong k^{n^2}$ is irreducible.

(b) Let $A \in \mathrm{Mat}(n \times n, k)$. Show that the centralizer

$$Z_{\mathrm{GL}(n,k)}(A) = \{g \in \mathrm{GL}(n, k) \mid gAg^{-1} = A\}$$

of A in $\mathrm{GL}(n, k)$ is connected.
Hint: Use an argument similar to that of part (a).

Exercise 17. Let G be a connected affine algebraic group, and $N \subset G$ a closed normal subgroup. Show: If N is finite, then N is contained in the center $Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$ of G.

Exercise 18. Let $\operatorname{char} k \neq 2$ and $V = k^n$. Let $\langle , \rangle : V \times V \to k$, $(v, w) \mapsto \sum v^i w^i$, be the standard symmetric bilinear form and O_n the group of all $g \in \mathrm{GL}(n, k)$ satisfying $\langle gv, gw \rangle = \langle v, w \rangle$ for all v, $w \in V$. Let $X = \{x \in V \mid \langle x, x \rangle = 1\}$.

(a) For $x \in X$ define $s_x : V \to V$, $v \mapsto v - 2\langle x, v \rangle x$. Show that s_x is in O_n.

(b) Fact (not to be proved): O_n is generated by the s_x, for $x \in X$. Deduce that $\mathrm{SO}_n = \{g \in \mathrm{O}_n \mid \det g = 1\}$ is connected.
Hint: Consider $X \times X \to \mathrm{GL}(n, k)$, $(x, y) \mapsto s_x s_y$.