

Exercise sheet 4

Solutions to be handed in by Wednesday 30th April 2008

Exercise 11.

- (a) Show that the special orthogonal group $\mathrm{SO}_2(\mathbb{C})$ and the multiplicative group $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ are isomorphic.
Hint: Consider the obvious operation of $\mathrm{SO}_2(\mathbb{C})$ on \mathbb{C}^2 and find a 1-dimensional $\mathrm{SO}_2(\mathbb{C})$ -stable subspace.
- (b) Find the connected components of the orthogonal group $\mathrm{O}_2(\mathbb{C})$.

Exercise 12. Let $f : X \rightarrow Y$ be a morphism of affine varieties and $f^* : k[Y] \rightarrow k[X]$ its comorphism. Show that

- (a) f^* is surjective $\Rightarrow f(X) \subset Y$ is closed;
- (b) f^* is injective $\Leftrightarrow f(X)$ is dense in Y ;
- (c) X is irreducible $\Rightarrow \overline{f(X)}$ is irreducible and $\dim \overline{f(X)} \leq \dim Y$.

Exercise 13.

- (a) If Y is a proper irreducible closed subvariety of an irreducible affine variety X , then $\dim Y < \dim X$.
- (b) If X and Y are irreducible affine varieties, then $\dim X \times Y = \dim X + \dim Y$.

Exercise 14. Let A and B be Hopf algebras.

- (a) Show that $A \otimes B$ becomes a Hopf algebra in a natural way.
- (b) If A and B arise from affine algebraic groups G and H , show that the comultiplication of $A \otimes B$ is the comorphism of the multiplication $(G \times H) \times (G \times H) \rightarrow G \times H$.