Algebraic Groups
Summer Semester 2008

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Exercise sheet 13

Solutions to be handed in on Wednesday 9nd July 2008

Problem 48. Let

$$E = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_4 = -x_1, \quad x_5 = -x_2, \quad x_3 = 0\}$$

and let $\varepsilon_1,\ \varepsilon_2 \in E^*$ be defined by $\varepsilon_i(x) = x_i$. Let $R = \{\pm\varepsilon_1 \pm \varepsilon_2, \pm\varepsilon_1, \pm\varepsilon_2\}$.

(a) Show that $R \subset E^*$ is a root system.
(b) Draw a picture of the 2-dimensional vector space E^* containing the roots and the corresponding reflection hyperplanes.
(c) Describe the Weyl group.

Problem 49. Let V be a 5 dimensional complex vector space with an ordered basis $e_1, e_2, e_0, e_{-1}, e_{-2}$. Define a symmetric bilinear form on V by declaring $(e_i, e_j) = 0$ $(i \neq -j)$, $(e_i, e_{-i}) = 1$ $(i \neq 0)$, $(e_0, e_0) = 2$.

(a) Let J be the matrix of this bilinear form with respect to the given basis. Compute J.
(b) Let $\mathfrak{g} = \{X \in \mathrm{Mat}(5 \times 5, \mathbb{C}) = \mathrm{End}(V) \mid X^T J + JX = 0\}$ be the Lie algebra $\mathfrak{so}(V) = \mathfrak{so}(5)$. Viewing elements of $\mathfrak{g}$ as block matrices, we can write

$$X = \begin{bmatrix} A & v & B \\ r & x & s \\ C & w & D \end{bmatrix},$$

where A, B, C, D are 2×2-matrices. Compute explicitly the conditions that the matrices $A, B, C, D, v, r, x, s, w$ must satisfy for X to belong to $\mathfrak{g}$.
(c) What is the dimension of $\mathfrak{g}$?
(d) Use without proof that the diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$. Compute the corresponding root system. (Use linear maps ε_1, ε_2 defined similarly as in Problem 48.)
(e) Find a Chevalley basis of $\mathfrak{g}$.

Problem 50. Describe all root systems of rank 2.