Algebraic Groups
Summer Semester 2008

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Exercise sheet 12

Solutions to be handed in on Wednesday 2nd July 2008

Problem 44.

(a) What is the center of $\mathrm{GL}_n(\mathbb{C})$?

Let $G := \mathrm{GL}_2(\mathbb{C})$.

(b) Consider the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}\, G)$. Restrict this representation to the subtorus $T \subset G$ of diagonal matrices. Then $\mathrm{Lie}\, G$ decomposes as

$$\mathrm{Lie}\, G = \bigoplus_{\chi \in X^*(T)} (\mathrm{Lie}\, G)_\chi,$$

where

$$(\mathrm{Lie}\, G)_\chi = \{X \in \mathrm{Lie}\, G \mid \mathrm{Ad}(t)(X) = \chi(t)X \text{ for all } t \in T\}$$

Determine the non-zero $(\mathrm{Lie}\, G)_\chi$.

(c) Similarly, restrict $\mathrm{ad} : \mathrm{Lie}\, G \to \mathrm{End}(\mathrm{Lie}\, G)$ to $\mathrm{Lie}\, T$ and find a decomposition

$$\mathrm{Lie}\, G = \bigoplus_{\lambda \in (\mathrm{Lie}\, T)^*} (\mathrm{Lie}\, G)_\lambda,$$

where

$$(\mathrm{Lie}\, G)_\lambda = \{X \in \mathrm{Lie}\, G \mid \mathrm{ad}(H)(X) = \lambda(H)X \text{ for all } H \in \mathrm{Lie}\, T\}.$$

(d) Determine all ideals of $\mathrm{Lie}\, G$. (An ideal of a Lie algebra is a vector subspace U of $\mathrm{Lie}\, G$ such that $[X, u] \in U$ for all $X \in \mathrm{Lie}\, G$ and $u \in U$.) Hint: Use the above decomposition.

Problem 45. Describe the quotient $\mathrm{GL}_{2n}(\mathbb{C})/\mathrm{Sp}_{2n}(\mathbb{C})$ using skew-symmetric matrices. Here $\mathrm{Sp}_{2n}(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid g^{\mathrm{t}}Jg = J\}$, where J is the $2n \times 2n$-matrix $\begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$.

Hint: If V is a finite dimensional vector space of dimension $2n$ with a non-degenerate skew-symmetric bilinear form ω, there is a basis $b_1, \dots, b_{2n}$ of V such that $\omega(b_i, b_j) = J_{ij}$ for all i, j.

Problem 46. Compute the dimension of the variety $\mathcal{F}$ of flags in $\mathbb{C}^n$ and show that there is an isomorphism $\mathrm{GL}_n(\mathbb{C})/B \xrightarrow{\sim} \mathcal{F}$ of varieties, where $B \subset \mathrm{GL}_n(\mathbb{C})$ is the subgroup of (invertible) upper triangular matrices.

Problem 47. Let G be an affine algebraic group and $N \subset G$ a closed subgroup. Show that the varieties $(G \times G)/(N \times N)$ and $G/N \times G/N$ are isomorphic.