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Exercise sheet 11

Solutions to be handed in on Wednesday 25th June 2008

Problem 40. Let $G = \mathrm{GL}(2, \mathbb{C})$ and $B \subset G$ the set of upper triangular matrices in G. Show that B is a closed subgroup and that the map

$$G \to \mathbb{P}^1\mathbb{C}, \quad g \mapsto [ge_1],$$

where e_1 is the first standard basis vector of $\mathbb{C}^2$, is the Chevalley quotient of G by B.

Problem 41. Let G, B be as in the previous problem. Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation of G. Show that $V^B = V^G$, i.e. any vector $v \in V$ fixed by B is fixed by the whole group G.

Hint: Apply the universal property of the categorical quotient $G \to G/B$ to the orbit map $g \mapsto \rho(g)(v)$ and use that that this quotient can be identified with the morphism $G \to \mathbb{P}^1\mathbb{C}$ from the previous problem.

Problem 42. Let V be a finite dimensional vector space, $f : V \to V$ a linear map and $r \in \mathbb{N}$. Consider the linear maps $\wedge^r(f)$ and $\Sigma^r(f) : \wedge^r V \to \wedge^r V$ given by

$$\wedge^r(f)(v_1 \wedge v_2 \wedge \dots v_r) := f(v_1) \wedge f(v_2) \wedge \dots f(v_r),$$

$$\Sigma^r(f)(v_1 \wedge v_2 \wedge \dots v_r) := \sum_{i=1}^{r} v_1 \wedge \dots \wedge f(v_i) \wedge \dots \wedge v_r.$$

Let $W \subset V$ be an r-dimensional subspace. Define $L := \wedge^r W$ and consider this as a one-dimensional subspace of $\wedge^r V$.

(a) If f is in $\mathrm{GL}(V)$ show that $f(W) = W$ if and only if $\wedge^r(f)(L) = L$.

(b) If f is in $\mathrm{End}(V)$ show that $f(W) \subset W$ if and only if $\Sigma^r(f)(L) \subset L$.

Problem 43. Let $\rho : G \to \mathrm{GL}(V)$ be a homomorphism of algebraic groups, where V is a finite dimensional vector space. We use the notation from the previous problem.

(a) Show that $\wedge^r \rho : G \to \mathrm{GL}(\wedge^r V)$, $g \mapsto \wedge^r(\rho(g))$, is a homomorphism of algebraic groups.

(b) Show that the differential $d\wedge^r \rho : \mathfrak{g} \to \mathrm{Lie}\,\mathrm{GL}(\wedge^r V) = \mathrm{End}(\wedge^r V)$ is given by $X \mapsto \Sigma^r(d\rho X)$.