Algebraic Groups
Summer Semester 2008

Catharina Stroppel
Olaf Schnürer

Exercise sheet 10

Solutions to be handed in on Wednesday 18th June 2008

Problem 36. Let $X = V$ be a finite dimensional k-vector space and $x \in X$.

(a) For $v \in V$, show that $D_v : k[X] \to k$, defined by
$$D_v(f) = \frac{d}{dt}|_{t=0}(f(x + tv)),$$
is a point derivation at x, i. e. $D_v \in \mathrm{Der}_k(k[X], k)$.

(b) Show that $v \mapsto D_v$ defines a canonical isomorphism $V \xrightarrow{\sim} T_xX$.

(c) If $U \subset X$ is an open subvariety and $x \in U$, show that $T_xU = T_xX$.

Problem 37. Let G be an affine algebraic group. Show that
$$(\mathrm{ad}\, X)(Y) = [X, Y]$$
for any X, $Y \in \mathrm{Lie}\, G = T_eG$.

Hint: If $\Delta(f) = \sum f_i \otimes g_i$, then $\tilde{D}(f) = \sum f_i D(g_i)$ and $[X, Y](f) = \sum X(f_i)Y(g_i) - Y(f_i)X(g_i)$.

Problem 38. Describe the Lie algebras of the following affine algebraic groups:

(a) $\mathrm{SL}(n, k) \subset \mathrm{GL}(n, k)$;

(b) upper triangular matrices $\subset \mathrm{GL}(n, k)$;

(c) strictly upper triangular matrices $\subset \mathrm{GL}(n, k)$.

Problem 39. Consider the affine algebraic group $\mathrm{PGL}(2, k)$ from exercise sheet 3, problem 8. Determine the Lie algebra of $\mathrm{PGL}(2, k)$ and describe the Lie algebra homomomorphism $\mathrm{Lie}\,\mathrm{SL}(2, k) \to \mathrm{Lie}\,\mathrm{PGL}(2, k)$. What happens in characteristic 2?