Quantum Groups
Winter Semester 2008/09

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Exercise Sheet 9

Solutions to be handed in on Monday 17th December 2008

Problem 30.

(a) Let A be a k-algebra and $p : A \to A$ a k-linear projection (i.e. $p^2 = p$). Assume that the kernel of p is a two-sided ideal.
Prove or disprove that p is an algebra homomorphism.

(b) Why is the Harish-Chandra map π from the lecture an algebra homomorphism?

Problem 31. Let $U = U_q(\mathfrak{sl}_2)$. Let q be a primitive l-th root of unity with l odd, $l \geq 3$

(a) Show that the center of U is generated by E^l, F^l and its intersection with U_0.
Hint: If $u \in U_m \setminus \{0\}$ is central then l divides m, and u is a product of a central element in U_0 with a power of E^l or a power of F^l.

(b) (Computational Challenge) The center of U is generated by E^l, F^l, K^l, K^{-l}, and C.

Problem 32. (**Harish-Chandra-Isomorphism**) Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and a let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra . Consider the space $\mathbb{C}[\mathfrak{h}^*] = U(\mathfrak{h})$ of polynomial functions on $\mathfrak{h}^*$, and let $\mathbb{C}[\mathfrak{h}^*]^W$ be the subalgebra of functions f satisfying $f(-\lambda) = f(\lambda)$ for all $\lambda \in \mathfrak{h}^*$. Show that the center Z of $U(\mathfrak{g})$ is isomorphic to $\mathbb{C}[\mathfrak{h}^*]^W$.

Hint: Adapt the methods of the lecture to the non-quantized situation: Take the standard basis (e, h, f) of $\mathfrak{g}$ and $\mathfrak{h} = \mathbb{C}h$. Define a projection $\pi : U(\mathfrak{g}) \to U(\mathfrak{h})$ and an automorphism γ_{-1} of $U(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$ by $f = f(\lambda) \mapsto f(\lambda - 1)$. Use the Casimir element $(h + 1)^2 + 4fe$.

Problem 33. Let $A = (a_{ij})$ be a real $l \times l$-matrix satisfying the following properties:

(a) A is indecomposable.

(b) $a_{ij} \leq 0$ for all $i \neq j$.

(c) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

For $u \in \mathbb{R}^l$ we write $u > 0$ ($u \geq 0$) if $u_i > 0$ ($u_i \geq 0$ respectively) for all $1 \leq i \leq l$. Similarly, $u < 0$ and $u \leq 0$. Show that one and only one of the following possibilities holds for both A and A^{t}:

(Fin) $\det A \neq 0$ and there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$.

(Aff) $\operatorname{rang} A = l - 1$ and there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$.

(Ind) there exists $u > 0$ such that $Au < 0$; if $Av \geq 0$ and $v \geq 0$ then $v = 0$.

Hints:

(a) Show (or use without proof): If B is a real $m \times s$ matrix for which there is no $u \geq 0$, $u \neq 0$, such that $B^{\mathrm{t}}u \geq 0$, then there exists $v > 0$ such that $Bv < 0$.

(b) Let A be as above. Then $Au \geq 0$ with $u \geq 0$ implies that either $u > 0$ or $u = 0$.

(c) Consider the convex cone

$$K_A = \{u \mid Au \geq 0\}.$$