Catharina Stroppel
Olaf Schnürer

Exercise Sheet 8

Solutions to be handed in on Monday 8th December 2008

Problem 26. (Baby Vermas) Suppose that q is a primitive l-th root of unity with l odd, $l \geq 3$. Let $S = k\langle E, K^{\pm 1}, F^l \rangle \subset U = U_q(\mathfrak{sl}_2)$. Let b, $\lambda \in k$ with $\lambda \neq 0$ and $k_{b,\lambda}$ the one-dimensional S-module given by

$$Ev = 0, \qquad Kv = \lambda v, \qquad F^l v = bv$$

for $v \in k = k_{b,\lambda}$. Recall the definition of the Baby Verma module $Z_b(\lambda) := U \otimes_S k_{b,\lambda}$.

(a) Show that $Z_b(\lambda)$ is a quotient of the Verma module $M(\lambda)$, more precisely

$$Z_b(\lambda) \cong M(\lambda)/U(m_l - bm_0),$$

where m_0, m_1, $\ldots$ is the usual basis of $M(\lambda)$.

(b) Show that $Z_b(\lambda)$ has a basis $\hat{m}_0, \ldots, \hat{m}_{l-1}$ such that

$$K\hat{m}_i = \lambda q^{-2i}\hat{m}_i,$$

$$F\hat{m}_i = \begin{cases} \hat{m}_{i+1}, & i < l-1 \\ b\hat{m}_0, & i = l-1, \end{cases}$$

$$E\hat{m}_i = \begin{cases} 0, & i = 0 \\ [i]\frac{\lambda q^{1-i} - \lambda^{-1}q^{i-1}}{q-q^{-1}}\hat{m}_{i-1} & i > 0. \end{cases}$$

Problem 27. Let k be an algebraically closed field of characteristic $p > 0$. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_n = \mathfrak{gl}_n(k)$ is called **restricted** if for all $x \in \mathfrak{g}$ we have $x^p \in \mathfrak{g}$ (where the p-th power is taken in $\mathfrak{gl}_n = \mathrm{M}_n(k)$).

Show: If G is a closed subgroup of the affine algebraic group $\mathrm{GL}_n(k)$, its Lie algebra is restricted.

Problem 28. Let $\mathfrak{g} \subset \mathfrak{gl}_n$ be a restricted Lie algebra (as in the previous exercise). Let $x \in \mathfrak{g}$. We write $x^{[p]}$ for the p-th power of x in $\mathrm{M}_n(k)$ and x^p for the p-th power of x in $U(\mathfrak{g})$.

(a) Show that $x^p - x^{[p]}$ is an element of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.
Hint: Given elements u, y in an associative algebra, we have $\mathrm{ad}(y)^p(u) = y^p u - uy^p$. Applying this to $\mathrm{M}_n(k)$ yields $\mathrm{ad}(x)^p = \mathrm{ad}(x^{[p]})$ on $\mathrm{M}_n(k)$, on $\mathfrak{g}$ and on $U(\mathfrak{g})$.

(b) For $p = 2$ and 3 (and if possible for arbitrary p) show that the map $\xi : \mathfrak{g} \to Z(\mathfrak{g})$, $x \mapsto x^p - x^{[p]}$, is semilinear, i. e. $\xi(x + y) = \xi(x) + \xi(y)$ and $\xi(ax) = a^p\xi(x)$ (for $a \in k$, x, $y \in \mathfrak{g}$).
Hint: In an arbitrary associative algebra, $(u+v)^p - u^p - v^p$ can be expressed in terms of iterated commutators. Apply this to $U(\mathfrak{g})$ and $\mathrm{M}_n(k)$.

(c) Let $Z_0(\mathfrak{g})$ be the subalgebra of $Z(\mathfrak{g})$ that is generated by all $\xi(x)$, $x \in \mathfrak{g}$. Let $x_1, \ldots, x_m$ be a basis of $\mathfrak{g}$. Show that the elements $\xi(x_1), \ldots, \xi(x_m)$ are algebraically independent generators of $Z_0(\mathfrak{g})$.

(d) Show that the algebra $U(\mathfrak{g})$ is free over $Z_0(\mathfrak{g})$ with basis

$$\{x_1^{a_1} x_2^{a_2} \ldots x_m^{a_m} \mid 0 \leq a_i \leq p-1 \text{ for all } i\}.$$

(e) Assume that k is an uncountable algebraically closed field. Deduce from the version of Schur's Lemma given in the next exercise that every irreducible representation of $U(\mathfrak{g})$ is finite dimensional.

Problem 29. (Schur's Lemma) Let k be an uncountable algebraically closed field (e. g. $\mathbb{C}$). Let A be a k-algebra of countable dimension. Let M be an irreducible A-module. Show that $k = \mathrm{End}_A(M)$.

Hint: Show that $\mathrm{End}_A(M)$ has countable dimension. On the other hand, the elements $(X - \lambda)^{-1}$, for $\lambda \in k$, are k-linear independent elements of the field $k(X)$ of rational functions in X.