Quantum Groups
Winter Semester 2008/09

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Exercise Sheet 7

Solutions to be handed in on Monday 1st December 2008

Problem 22.

(a) **Tensor identity:** Let $B \to A$ be a homomorphism of Hopf algebras over k, C a B-module and E an A-module. Show that

$$A \otimes_B (C \otimes_k \operatorname{res}_B^A E) \to (A \otimes_B C) \otimes_k E$$
$$a \otimes (c \otimes e) \mapsto \sum_{(a)} a_{(1)} \otimes c \otimes a_{(2)} e$$

is a well-defined isomorphism of A-modules. (The inverse is $(a \otimes c) \otimes e \mapsto \sum_{(a)} a_{(1)} \otimes c \otimes S(a_{(2)})e$.)

(b) Let $\mathfrak{b}$ be a subalgebra of a Lie algebra $\mathfrak{g}$, $\mathbb{C}_\lambda$ a representation of $\mathfrak{b}$ and E a representation of $\mathfrak{g}$. Show that

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes_k E) \xrightarrow{\sim} (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda) \otimes_k E$$

as $U(\mathfrak{g})$-modules.

(c) Consider $U = U_q(\mathfrak{sl}_2)$ and let $M(\lambda)$ be a Verma module, for $\lambda \in k \setminus \{0\}$. Describe $M(\lambda) \otimes_k V$ where V is the 2-dimensional irreducible representation of type I of U (we assume that q is not a root of unity).

Problem 23. Let $U = U_q(\mathfrak{sl}_2)$, where $q \in k$ is not a root of unity and $\operatorname{char} k \neq 2$. Let V_i be the irreducible $(i+1)$-dimensional U-module of type I.

(a) Are the following true as isomorphisms of U-modules?

$$V_i \otimes_k V_0 \cong V_i \cong V_0 \otimes_k V_i$$
$$V_1 \otimes_k V_2 \cong V_2 \otimes_k V_1$$
$$V_2 \otimes_k V_3 \cong V_3 \otimes_k V_2$$

If so, can you find an isomorphism?

(b) Decompose $V_i \otimes_k V_j$ into a direct sum of irreducible modules (Clebsch-Gordan formula).

Problem 24. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be a Cartan and a Borel subalgebra. Any $\lambda \in \mathfrak{h}^*$ extends uniquely to a Lie algebra homomorphism $\mathfrak{b} \to \mathbb{C}$, i.e. to a one dimensional representation $\mathbb{C}_\lambda$ of $\mathfrak{b}$ (since the extension has to vanish on the nilradical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$). The Verma module $M(\lambda)$ is defined by $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$.

(a) Show: $M(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} M(\lambda)_\mu$, i.e. $M(\lambda)$ decomposes into weight spaces with respect to the action of $\mathfrak{h}$.

(b) Give an explicit formula for $\dim M(\lambda)_\mu$ in terms of the combinatorics of roots.

Problem 25. Let $U = U_q(\mathfrak{sl}_2)$ and $\lambda,\ \lambda' \in \{q^n \mid n \in \mathbb{Z}\}$ (and q not a root of unity). Assume we have a short exact sequence (i. e. f injective, g surjective and $\operatorname{im} f = \ker g$) of U-modules of the form

$$0 \to M(\lambda') \xrightarrow{f} M \xrightarrow{g} M(\lambda) \to 0.$$

(a) For which pairs (λ, λ') do we have $M \cong M(\lambda') \oplus M(\lambda)$ automatically?
Hint: Use the Casimir.

(b) Assume $M = \bigoplus M_\mu$ decomposes into weight spaces. Can you say more in this case?

(c) Give examples of U-modules M which fit into a short exact sequence of the form

$$0 \to M(\lambda) \to M \to M(\lambda) \to 0$$

but are not isomorphic to $M(\lambda) \oplus M(\lambda)$.