Quantum Groups
Winter Semester 2008/09

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Exercise Sheet 6

Solutions to be handed in on Monday 24th November 2008

Problem 19. Prove the following theorem (which is also called the **Cartier-Gabriel-Milnor-Moore theorem**):

Let H be a graded connected cocommutative Hopf algebra over an algebraically closed field of characteristic zero. Then H is isomorphic (as a graded Hopf algebra) to the universal enveloping algebra of its primitive elements.

Problem 20. Let $\mathfrak{g}$ be a complex Lie algebra. For simplicity let us assume that it is finite dimensional. Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{g}$. Recall that the Poincare-Birkhoff-Witt theorem says that the monomials

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

for $i_j \in \mathbb{Z}_{\geq 0}$ form a basis of the universal enveloping algebra. Give a proof of this theorem using the Diamond Lemma.

Problem 21. Let V be a finite dimensional vector space over an infinite field (e. g. an algebraically closed field) of characteristic zero.

(a) Show that $\mathrm{Sym}(V^*) = k[V]$, where $k[V]$ is the algebra of polynomial functions on V.

(b) Show that the composition $\Gamma(V)_n \to T(V)_n \to \mathrm{Sym}(V)_n$ is an isomorphism of vector spaces.

(c) Use (b) to show that the map $\phi : (\Gamma(V)_n)^* \to \mathrm{Sym}(V^*)_n$, defined by $\phi(f)(v) = f(v^{\otimes n})$, defines an isomorphism of vector spaces; in particular, for every polynomial map $p : V \to k$ there is a unique linear map $f : \Gamma(V)_n \to k$ such that $p(v) = f(v^{\otimes n})$ for all $v \in V$.

(d) Let $\mathfrak{g} = \oplus_{i \geq 1} \mathfrak{g}_i$ be a (strictly positively) graded vector space with finite dimensional $\mathfrak{g}_i$'s. Show that the graded dual of $\Gamma := \Gamma(\mathfrak{g})$ is isomorphic to $\mathrm{Sym}(\mathfrak{g}^{\circledast})$ as a graded vector space, where $\mathfrak{g}^{\circledast}$ is the graded dual of $\mathfrak{g}$.

Hint: Define a bigrading on Γ by $\Gamma_{n,p} = \Gamma_n \cap \Gamma^p$, where

$$\Gamma_n = \Gamma \cap T(\mathfrak{g})_n,$$
$$\Gamma^p = \Gamma \cap \Big(\bigoplus_{r \in \mathbb{N}} \bigoplus_{\substack{i_1, \ldots, i_r \\ i_1 + \cdots + i_r = p}} \mathfrak{g}_{i_1} \otimes \cdots \otimes \mathfrak{g}_{i_r} \Big).$$

Note that Γ^p is finite dimensional. Take the graded dual of Γ with respect to the p-grading, i. e. $\Gamma^{\circledast} = \bigoplus_p (\Gamma^p)^*$. Similarly, define a bigrading on $\mathrm{Sym}(\mathfrak{g})$ and prove using (c) that $\Gamma^*_{n,p} \xrightarrow{\sim} \mathrm{Sym}(\mathfrak{g}^{\circledast})_{n,p}$.

(e) Deduce the last step in the proof (from the lecture) of the Milnor-Moore theorem.