Quantum Groups
Winter Semester 2008/09

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Exercise sheet 5

Solutions to be handed in on Monday 17th November 2008

Problem 15. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k.

(a) Show: $U(\mathfrak{g})$ is an example of a cocommutative, conilpotent Hopf-algebra. (Of course you are not allowed to use the Classification Theorem of cocommutative (conilpotent) Hopf algebras).

Assume now that k has characteristic zero.

(b) Choose a basis $\{x_1, \cdots, x_n\}$ of $\mathfrak{g}$. Then the elements $Z_\alpha = \prod_{i=1}^n \frac{1}{\alpha_i!} x_i^{\alpha_i}$ for $\alpha \in \mathbb{Z}_{\geq 0}^n$ form a basis of $U(\mathfrak{g})$.

(c) Show: In this basis the comultiplication has the form

$$\Delta(Z_\alpha) = \sum_{\beta+\gamma=\alpha} Z_\beta \otimes Z_\gamma.$$

(d) Deduce: The primitive elements in $U(\mathfrak{g})$ are exactly the elements of $\mathfrak{g}$.

Remark: The assumptions of finite-dimensionality is superfluous.

Problem 16. Prove the **Theorem of Milnor-Moore**: Let $A = \bigoplus_{n\geq 0} A_n$ be a graded Hopf algebra over a field k of characteristic zero. Asume that A is connected (i. e. $A_0 = k\cdot 1$) and that the product in A is commutative. Then A is a free commutative algebra (i. e. a polynomial algebra) generated by homogeneous elements.

Hint: A graded Hopf algebra is a Hopf algebra A with a vector space decomposition $A = \bigoplus_{n\geq 0} A_n$ such that the multiplication satisfies $m(A_i \otimes A_j) \subseteq A_{i+j}$, and $\Delta(A_n) \subseteq \bigoplus_{i+j=n} A_i \otimes A_j$.

For the proof follow the following steps:

(a) Define, as in the lecture, $\Psi_n : A \to A$, for $n \geq 1$, as the n-fold convolution of the identity.

(b) Decompose $A = \bigoplus_{p\geq 0} \pi_p(A)$ where $\pi_p(A) = \{a \in A \mid \Psi_n(a) = n^p a \text{ for all } n \geq 1\}$.

(c) Show: There is a well-defined algebra isomorphism

$$\Theta : \mathrm{Sym}(\pi_1(A)) \to A$$

from the symmetric algebra of $\pi_1(A)$ to A.

Problem 17. Let k be a field of characteristic zero. Show: The characters of the symmetric groups form a graded Hopf algebra $\mathrm{Ch} = \bigoplus_{n\geq 0} \mathrm{Ch}_n$ isomorphic to a polynomial ring.
Proceed as follows:

(a) Let Ch_n be the vector space of all functions $S_n \to k$ constant on conjugacy classes.
(b) Show that $\mathrm{Ch}_p \otimes \mathrm{Ch}_q$ can be identified with the space of functions $f : S_p \times S_q \to k$ that are constant on conjugacy classes.
(c) View $S_p \times S_q$ as a subgroup of S_{p+q}. This yields a restriction map
$$\Delta_{p,q} : \mathrm{Ch}_{p+q} \to \mathrm{Ch}_p \otimes \mathrm{Ch}_q\,.$$
Use this to define a comultiplication on Ch.
(d) Define on each Ch_n a non-degenerate bilinear form and dualize the comultiplication into a multiplication. Extend this to a Hopf algebra structure. Apply Milnor-Moore.

Problem 18.

(a) Let A be a commutative finite dimensional algebra over an algebraically closed field k. Then A is isomorphic to a direct product of algebras A_i, $1 \leq i \leq n$, where each A_i is a local ring whose maximal ideal $\mathfrak{m}_i$ is nilpotent (i.e. $\mathfrak{m}_i^N = 0$ for some big enough N). (This is a special case of the so-called structure theorem for semilocal rings.)
(b) Now assume additionally that $A = C^*$ is the dual of a coalgebra. Then the algebra homomorphisms from A to k are in bijection to the group-like elements in C.