Quantum Groups
Winter Semester 2008/09

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Exercise sheet 4

Solutions to be handed in on Monday 10th November 2008

Problem 12. Let G be a compact group.

(a) Let V be a simple (finite-dimensional) complex G-module and $f \in \mathrm{End}_{\mathbb{C}}(V)$. Using Schur's lemma show that

$$\int_G g \circ f \circ g^{-1}\, dg = \frac{1}{\dim_{\mathbb{C}} V} \operatorname{tr}(f) \operatorname{id}_V .$$

(b) Deduce that for $\lambda \in V^*$ and $v \in V$

$$\int_G \lambda(gf(g^{-1}v))\, dg = \frac{1}{\dim_{\mathbb{C}} V} \operatorname{tr}(f)\lambda(v).$$

(c) Show that the irreducible characters span a dense subspace in the space of all continuous class functions.

Hint: Approximate a given class function by a representative function using Peter-Weyl. Average this function and use the above results.

Problem 13. Let G be a compact group and T a compact operator on a Hilbert G-module $\mathcal{H}$. Then $\widetilde{T} = \int_G g^{-1} T g\, dg$ is also compact.

Hint: Let B be the closed unit ball of $\mathcal{H}$. Then $A := \overline{TGB} = \overline{TB}$ and GA are compact. Let K be the closed convex hull of GA. It is compact (the closed convex hull of a precompact set in a Banach space is compact). For arbitrary $y \in \mathcal{H}$ let $H_y := \{z \in \mathcal{H} \mid \mathrm{Re}(z, y) \leq 1\}$ be the closed real half-space defined by y. If $K \subset H_y$ show that $\widetilde{T}B \subset H_y$. Deduce from Hahn-Banach that $\widetilde{T}B \subset K$.

Problem 14. Show that every Hilbert G-module $\mathcal{H}$ for a compact group G is a Hilbert space orthogonal sum of finite dimensional simple G-submodules.

Hint: Use Zorn's Lemma to find a maximal family $\mathcal{F} = \{E_i \mid i \in I\}$ of finite dimensional simple submodules such that $i \neq j$ implies $E_i \perp E_j$ for all $i,\, j \in I$.