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Winter Semester 2008/09 | Olaf Schnürer

Exercise sheet 3

Solutions to be handed in on Monday 3rd November 2008

Problem 7. Provide at least two examples of bialgebras which cannot be equipped with an antipode turning them into Hopf algebras.

Problem 8. Let C be a coalgebra and A an algebra, both over a field k.

(a) Show: There is an algebra structure on $\mathrm{Hom}_k(C, A)$ with the convolution product $*$ as multiplication.
(b) Show that the group like-elements $\mathcal{G}(C)$ (without zero) of C are k-linearly independent in C. Let $k\mathcal{G}(C) \subset C$ be the subspace generated by these elements. Show that $k\mathcal{G}(C)$ forms a subcoalgebra of C.
(c) Describe the algebra structure on $\mathrm{Hom}_k(k\mathcal{G}(C), k) = \mathrm{Maps}(\mathcal{G}(C), k)$ from (a) explicitly.

Problem 9. Let H be a Hopf algebra.

(a) Show: The antipode of H is an antihomomorphism of algebras.
(b) Let M be an H-module and consider the natural homomorphisms of vector spaces (M finite dimensional for the last row)

$$\begin{array}{lll} M \to (M^*)^*, & & \\ M^* \otimes M \to k, & M \otimes M^* \to k & \text{(evaluation)} \\ k \to M \otimes M^* & k \to M^* \otimes M & \end{array}$$

Which are homomorphisms of H-modules? Which conditions should H satisfy such that they are all homomorphisms of H-modules?

Problem 10. (Sweedler's Hopf algebra) Consider the algebra A generated by g and x with relations $g^2 = 1$, $x^2 = 0$, $gxg = -x$. Show that there is a non-commutative, non-cocommutative Hopf algebra structure on A such that g is group-like and x is primitive. Compute the dimension of A.

Remark: This is the non-commutative, non-cocommutative Hopf algebra of smallest dimension.

Problem 11. Let G be a group and $k[G]$ its group algebra. Its dual $k[G]^* = (k[G])^* = \mathrm{Maps}(G, k)$ is a $k[G]$-bimodule via

$$(x.f)(z) = f(zx),$$
$$(f.y)(z) = f(yz),$$

for x, y, $z \in G$. Let $k[G]^\circ \subset k[G]$ be the subspace of representative functions.

(a) Define $\pi : k[G]^* \otimes k[G]^* \to k[G \times G]^*$ by $(\pi(f \otimes g))(x, y) = f(x)g(y)$ for x, $y \in G$. Show that π is injective.
(b) Define $\delta : k[G]^* \to k[G \times G]^*$ by $(\delta f)(x, y) = f(xy)$ for x, $y \in G$. Show that

$$\delta(k[G]^\circ) \subset \pi(k[G]^\circ \otimes k[G]^\circ).$$

(c) Show that $k[G]^\circ$ has a Hopf algebra structure with comultiplication $\Delta = \pi^{-1} \circ \delta$.