Quantum Groups
Winter Semester 2008/09

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Exercise sheet 2

Solutions to be handed in on Wednesday 29th October 2008

Problem 3. (Ore localization) Let R be a ring and $S \subset R$ a multiplicative subset (i. e. $1 \in S$ and $a, b \in S \Rightarrow ab \in S$). Assume that S is a **right denominator set**, i. e.

(i) $\forall a \in R,\ s \in S$: $aS \cap sR \neq \emptyset$, and
(ii) $\forall a \in R,\ s \in S$: $sa = 0 \Rightarrow \exists t \in S$: $at = 0$.

Show that

$$(a, s) \sim (a', s') \text{ if } au = a'u' \text{ and } su = s'u' \in S \text{ for some } u,\, u' \in R$$

defines an equivalence relation on $R \times S$. Denote the equivalence class of (a, s) by a/s and let $R[S^{-1}]$ be the set of equivalence classes. Equip $R[S^{-1}]$ with a ring structure such that $R \to R[S^{-1}]$, $r \mapsto r/1$, is universal among all ring homomorphisms $R \to A$ inverting all elements of S.

(a) Show: If S is generated by a finite set and R is noetherian, then $R[S^{-1}]$ is noetherian.
(b) Consider $\mathcal{O}_q(k^n)$ and let X be the multiplicative set generated by $X_1, \ldots, X_n$. Show that X is a right denominator set and that

$$\mathcal{O}_q(k^n)[X^{-1}] \xrightarrow{\sim} \mathcal{O}_q((k^\times)^n).$$

Problem 4. Show:

(a) $\mathcal{O}_q(\mathrm{M}(2, k))$ is a bialgebra (with the formulas from the lecture). Is it (co)commutative?
(b) Its bialgebra structure induces a bialgebra structure on $\mathcal{O}_q(\mathrm{SL}(2, k))$ which can be extended to a Hopf algebra structure.
(c) $\mathcal{O}_q(\mathrm{GL}(2, k))$ is a Hopf algebra.
(d) $\mathcal{O}_q(k^2)$ is a $\mathcal{O}_q(\mathrm{SL}(2, k))$-comodule algebra.

Problem 5. Show: Any Hopf algebra H is a left and right H-module algebra via

$$h.m = \sum\nolimits_{(h)} h' m S(h'') \text{ and}$$
$$m.h = \sum\nolimits_{(h)} S(h') m h''$$

respectively, where h is in H and m in the H-module H. Here S is the antipode and we use the Sweedler notation $\Delta(h) = \sum_{(h)} h' \otimes h''$.

Problem 6. Let $G = (k, +)$ be the additive group and $\mathfrak{g}$ its Lie algebra. Find a natural Hopf algebra structure on

$$L := \{f \in \mathcal{O}(G)^* = \mathrm{Hom}_k(\mathcal{O}(G), k) \mid f(\mathfrak{m}^n) = 0 \text{ for } n \gg 0\},$$

where $\mathfrak{m} = \{f \in \mathcal{O}(G) \mid f(1) = 0\}$ is the so called augmentation ideal, and an isomorphism of Hopf algebras

$$U(\mathfrak{g}) \xrightarrow{\sim} L.$$

Show the same statement for an arbitrary affine algebraic group.