Quantum Groups
Winter Semester 2008/09

Catharina Stroppel
Olaf Schnürer

Exercise Sheet 13

Solutions to be handed in on Monday 2nd February 2009

Problem 44. Let $\mathcal{B}(m) = \{F^{(i)}u \mid 0 \leq i \leq m\}$ be the crystal graph of the $(m+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module $L(m)$ with highest weight vector u and let $\mathcal{B}(\infty) = \{F^{(i)}v_0 \mid i \geq 0\}$ be the crystal graph of the Verma module $M(0)$. Show: The map $\Psi : \mathcal{B}(m) \to \mathcal{B}(\infty) \otimes T_m$, $f^{(i)}u \mapsto f^{(i)}v_0 \otimes t_m$ $(0 \leq i \leq m)$, is an embedding of $U_q(\mathfrak{sl}_2)$-crystals, but this crystal morphism is not strict.

Problem 45. Let $\lambda \in X^+$ be a dominant integral weight and $\mathcal{B}(\lambda)$ the crystal graph of the irreducible highest weight $U_q(\mathfrak{g})$-module $L(\lambda)$. Suppose that $\mathcal{B}$ is a connected $U_q(\mathfrak{g})$-crystal. If there exists a strict crystal morphism $\Psi : \mathcal{B}(\lambda) \to \mathcal{B}$ such that $\Psi(\mathcal{B}(\lambda)) \subset \mathcal{B}$, then Ψ is a crystal isomorphism.

Problem 46. Let V be the three-dimensional $U_q(\mathfrak{sl}_3)$-module that is the quantized version of the natural representation of $\mathfrak{sl}_3$. Let $(\mathcal{L}, \mathcal{B})$ be its crystal basis. Draw the crystal graph $\mathcal{B} \otimes \mathcal{B}$ and show that $V \otimes V \cong L(2\epsilon_1) \oplus L(\epsilon_1 + \epsilon_2)$ as $U_q(\mathfrak{sl}_3)$-modules.

Problem 47. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be normal crystals. Show that any injective morphism $f : \mathcal{B}_1 \to \mathcal{B}_2$ of crystals is strict.

Problem 48. Let $\lambda \in X^+$ be a miniscule dominant weight. Let $\pi_\lambda : [0, 1] \to X_{\mathbb{R}}$, $t \mapsto t\lambda$, be the path connecting 0 and λ in a straight line. Compute what is generated by π_λ under the action of the Littelmann root operators e_α, f_α. Compare it with the character formula for $L(\lambda)$.