Quantum Groups
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Exercise Sheet 12

Solutions to be handed in on Monday 19th January 2009

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ a Cartan and a Borel subalgebra. Let $\lambda \in \mathfrak{h}^*$ and $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ the (non-quantized) Verma module with highest weight λ (cf. Problem 24).

Problem 40. Let $\mathbb{Z}\mathfrak{h}^*$ be the group ring of the additive group $\mathfrak{h}^*$. We write e^λ if we consider $\lambda \in \mathfrak{h}^*$ as an element of $\mathbb{Z}\mathfrak{h}^*$. The $(e^\lambda)_{\lambda \in \mathfrak{h}^*}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}\mathfrak{h}^*$, multiplication of basis elements is given by $e^\lambda e^\mu = e^{\lambda+\mu}$. If V is a finite dimensional representation of $\mathfrak{g}$ we define its **character** $\operatorname{ch} V \in \mathbb{Z}\mathfrak{h}^*$ by

$$\operatorname{ch} V := \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e^\lambda.$$

(Note that V decomposes into a direct sum of its weight spaces.)

(a) If V and W are finite dimensional representations of $\mathfrak{g}$, then $\operatorname{ch}(V \oplus W) = \operatorname{ch}(V) + \operatorname{ch}(W)$ and $\operatorname{ch}(V \otimes W) = \operatorname{ch}(V)\operatorname{ch}(W)$.

(b) Let $\operatorname{Maps}(\mathfrak{h}^*, \mathbb{Z})$ be the set of all maps from $\mathfrak{h}^*$ to $\mathbb{Z}$. If f is such a function, we denote it by $f = \sum_{\lambda \in \mathfrak{h}^*} f(\lambda) e^\lambda$. This identifies $\mathbb{Z}\mathfrak{h}^*$ with the subset of $\operatorname{Maps}(\mathfrak{h}^*, \mathbb{Z})$ whose elements have finite support. Let $\widehat{\mathbb{Z}\mathfrak{h}^*} \subset \operatorname{Maps}(\mathfrak{h}^*, \mathbb{Z})$ be the subset whose elements have support contained in a finite union of sets of the form $\mu - \mathbb{N}\Phi^+$ (where Φ^+ is the set of positive roots corresponding to $\mathfrak{b}$). Show that the ring structure of $\mathbb{Z}\mathfrak{h}^*$ can be naturally extended to $\widehat{\mathbb{Z}\mathfrak{h}^*}$.

(c) If V is a representation of $\mathfrak{g}$ with finite dimensional weight spaces, it gives rise to an element $\operatorname{ch} V = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu \in \operatorname{Maps}(\mathfrak{h}^*, \mathbb{Z})$. Show that $\operatorname{ch} M(\lambda) \in \widehat{\mathbb{Z}\mathfrak{h}^*}$.

(d) Show that

$$\operatorname{ch} M(\lambda) = \sum_{\nu \in \mathfrak{h}^*} \mathcal{P}(\lambda - \nu) e^\nu,$$

where $\mathcal{P}$ is Kostant's partition function, and

$$\operatorname{ch} M(\lambda) = e^\lambda \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha} + e^{-2\alpha} + \dots) = e^\lambda \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}. \tag{0.1}$$

Problem 41. Consider $\mathfrak{g} = \mathfrak{sl}_n$ with the usual choice of Borel and Cartan subalgebra. Compute the character and the highest weight of

- the natural representation on $\mathbb{C}^n$ and
- the adjoint representation ad on $\mathfrak{sl}_n$.

Compute now (in the quantized situation) the character (i. e. the dimensions of all weight spaces) of the $U_q(\mathfrak{sl}_3)$-module $\widetilde{L}(\lambda)$, for λ the two weights computed above.

Problem 42. We use the notation of Problem 40. For $\lambda \in \mathfrak{h}^*$ let $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$. Prove **Weyl's character formula**: For $\lambda \in X^+$ we have

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w\rho}},$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ is the half-sum of all positive roots.

Hint: Proceed as follows.

(a) Use without proof: For $\lambda \in \mathfrak{h}^*$, the Verma module $M(\lambda)$ has a finite filtration with subquotients $L(\mu)$, where $\mu \in W \cdot \lambda$. Here the dot-Operation of W is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

(b) Show that

$$\operatorname{ch} M(\lambda) = \sum_{\substack{\mu \in W \cdot \lambda \\ \mu \leq \lambda}} b_\mu \operatorname{ch} L(\mu)$$

for some $b_\mu \in \mathbb{Z}$, $b_\lambda = 1$.

(c) Deduce that

$$\operatorname{ch} L(\lambda) = \sum_{\mu \in W \cdot \lambda} a_\mu \operatorname{ch} M(\mu) \tag{0.2}$$

with $a_\mu \in \mathbb{Z}$, $a_\lambda = 1$.

(d) Let $d = \sum_{w \in W} (-1)^{l(w)} e^{w\rho}$, the denominator in Weyl's formula. You may assume the first equality in **Weyl's denominator identity**:

$$d = e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}). \tag{0.3}$$

(e) Multiply (0.2) by d and use (0.1) and (0.3) to obtain

$$d \operatorname{ch} L(\lambda) = \sum_{w \in W} c_w (-1)^{l(w)} e^{w(\lambda+\rho)}$$

for some $c_w \in \mathbb{Z}$. If λ is in X^+, the left hand side is W-anti-invariant. Deduce Weyl's character formula.

Problem 43. Let M_1 and M_2 be finite dimensional U-modules, where $k = \mathbb{Q}(q)$ and $U = U_q(\mathfrak{g})$. Let $\mathcal{M}_1 \subset M_1$ and $\mathcal{M}_2 \subset M_2$ be A-submodules.

(a) Show: $\mathcal{M}_1 \oplus \mathcal{M}_2$ is an admissible lattice in $M_1 \oplus M_2$ if and only if $\mathcal{M}_1 \subset M_1$ and $\mathcal{M}_2 \subset M_2$ are both admissible lattices.

(b) Let $\mathcal{B}_1 \subset \mathcal{M}_1 / q\mathcal{M}_1$ and $\mathcal{B}_2 \subset \mathcal{M}_2 / q\mathcal{M}_2$ be subsets. Show:

$$(\mathcal{M}_1 \oplus \mathcal{M}_2, (\mathcal{B}_1 \times 0) \cup (0 \times \mathcal{B}_2))$$

is a crystal base of $M_1 \oplus M_2$ if and only if $(\mathcal{M}_1, \mathcal{B}_1)$ is a crystal base of M_1 and $(\mathcal{M}_2, \mathcal{B}_2)$ is a crystal base of M_2.