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Winter Semester 2008/09 | Olaf Schnürer

Exercise Sheet 11

Solutions to be handed in on Monday 12th January 2009

Let $U = U_q(\mathfrak{g})$ with q not a root of unity.

Problem 37. Verify the following claim: To show that every finite dimensional U-module is semisimple it is enough to show that every short exact sequence $0 \to L(\lambda) \to M \to L(\mu) \to 0$ of U-modules with λ, $\mu \in X^+$ splits.
Hint: Use the long exact Ext-sequences.

Problem 38.

(a) Show: Every finite dimensional irreducible U-module (of type I) is a direct summand of some $L(\omega_{i_1}) \otimes L(\omega_{i_2}) \otimes \cdots \otimes L(\omega_{i_r})$ for $i_j \in \{1, 2, \ldots, l\}$ (for $r = 0$ this tensor product is the trivial representation $L(0)$).

(b) A dominant weight $\lambda \in X^+$ is miniscule if $\lambda \neq 0$ and $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$. Show that a dominant weight λ is miniscule if and only if there is no $\mu \in X^+$ with $\mu < \lambda$.

(c) Determine all miniscule dominant weights for type A_2 and B_2.

(d) Let $\lambda \in X^+$. Show: $L(\lambda)^* \cong L(-w_0\lambda)$ where $w_0 \in W$ is the longest element.
Deduce that the direct sum M of all $L(\lambda)$, for λ miniscule dominant, is selfdual, i. e. $M \cong M^*$.

(e) Show: If λ is miniscule dominant, then $X(L(\lambda)) = W\lambda$ and the weight spaces are all 1-dimensional.

(f) Let λ be miniscule dominant. Consider a vector space V with basis x_μ, $\mu \in W\lambda$. Show that

$$
\begin{aligned}
&K_\alpha x_\mu = q^{(\mu,\alpha)} x_\mu && K_\alpha^{-1} x_\mu = q^{-(\mu,\alpha)} x_\mu && \text{and} \\
&E_\alpha x_\mu = 0, && F_\alpha x_\mu = 0, && \text{if } \langle \mu, \alpha^\vee \rangle = 0; \\
&E_\alpha x_\mu = 0, && F_\alpha x_\mu = x_{\mu-\alpha}, && \text{if } \langle \mu, \alpha^\vee \rangle = 1; \\
&E_\alpha x_\mu = x_{\mu+\alpha}, && F_\alpha x_\mu = 0, && \text{if } \langle \mu, \alpha^\vee \rangle = -1;
\end{aligned}
$$

($\alpha \in \Pi$, $\mu \in W\lambda$) defines a U-module structure on V such that $V \cong L(\lambda)$.

(g) Show for $\mathfrak{g}$ of type A_n: The $L(\lambda)$, λ miniscule dominant, include the quantum analogue of the "natural" representation of $\mathfrak{g}$, i. e. the representation of $\mathfrak{sl}_{n+1}$ on k^{n+1}.
Remark: This is also true in types C_n and D_n, but not in type B_n.

(h) Assume $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Show: All roots have the same length (the length of a root α is $\sqrt{(\alpha, \alpha)}$ where $(\cdot, \cdot)$ is the bilinear extension of $(\alpha_i, \alpha_j) = a_{ij} d_j$). The (quantum analogue) of the adjoint representation is $L(\lambda)$, where λ is the largest (short) root (with respect to the partial ordering on X).

Remark: The modules $L(\lambda)$ for λ miniscule dominant are important, since they are quite easy to describe and every $L(\omega_i)$ occurs in a tensor product of $L(\lambda)$'s, where the λ's are miniscule dominant weights or largest short roots.

Problem 39. Let $u \in U$. Show: If u annihilates all finite dimensional U-modules, then $u = 0$.

Hint: Prove this by verifying the following:

(1) Given $\lambda = \sum_{\alpha\in\Pi} n_\alpha \omega_\alpha \in X^+$ and $\nu = \sum_{\alpha\in\Pi} m_\alpha \alpha \in \mathbb{Z}_{\geq 0}\Phi$ we have surjections

$$U^-_{-\nu} \twoheadrightarrow \tilde{L}(\lambda)_{\lambda-\nu}, \qquad y \mapsto y v_\lambda,$$
$$U^+_{\nu} \twoheadrightarrow ({}^\omega\tilde{L}(\lambda))_{-\lambda+\nu}, \qquad x \mapsto x v'_\lambda,$$

which are bijections if $m_\alpha \leq n_\alpha$ for all $\alpha \in \Pi$.

(2) Choose bases $(x_i)_i$ of U^+ and $(y_j)_j$ of U^- consisting of weight vectors, say $x_i \in U^+_{\nu(i)}$ and $y_j \in U^-_{-\nu'(j)}$ with $\nu(i)$ and $\nu'(j)$ in $\mathbb{Z}_{\geq 0}\Phi$ and write

$$u = \sum_j \sum_{\mu\in\mathbb{Z}\Phi} \sum_i a_{j,\mu,i} y_j K_\mu x_i$$

and

(a) Calculate the action of x_i on $v_\lambda \otimes v'_{\lambda'} \in \tilde{L}(\lambda) \otimes {}^\omega\tilde{L}(\lambda')$ (for λ, λ' dominant).

(b) Then calculate the action of $K_\mu x_i$ on $v_\lambda \otimes v'_{\lambda'}$.

(c) Choose $\nu_0 \in \mathbb{Z}\Phi$ maximal among the weights ν such that there exist i, μ, j, with $a_{j,\mu,i} \neq 0$ and $\nu = \nu(i)$.
Compute the projection of $u(v_\lambda \otimes v'_{\lambda'})$ onto $\tilde{L}(\lambda) \otimes ({}^\omega\tilde{L}(\lambda'))_{-\lambda'+\nu_0}$.

(d) Use (1) to deduce that for λ' big enough the $x_i v'_{\lambda'}$ for $\nu(i) = \nu_0$ are linearly independent.

(e) Use an analogue of (2)d for the $y_j v_\lambda$'s to deduce that

$$u(v_\lambda \otimes v'_{\lambda'}) = 0 \Longrightarrow \sum_\mu a_{j,\mu,i} q^{(\mu,\nu_0-\lambda')} q^{(\mu,\lambda)} = 0$$

for all i, j with $\nu(i) = \nu_0$ and λ, λ' big enough.

(f) Use (a variant of) Artin's theorem on the linear independence of characters to show that $u = 0$.