Quantum Groups
Winter Semester 2008/09

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Exercise Sheet 10

Solutions to be handed in on Monday 5th January 2009

Problem 34. Let $w \in W$ and $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ an expression as a product of simple reflections (not necessarily reduced).

(a) Assume that $s_{i_1} s_{i_2} \cdots s_{i_{r-1}}(\alpha_{i_r})$ is negative. Show that there is an index $1 \leq k < r$ such that

$$s_{i_1} s_{i_2} \cdots s_{i_r} = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{i_{k+1}} \ldots s_{i_{r-1}},$$

i. e. the above expression was not reduced.

Hint: Let t be minimal such that $s_{i_{t+1}} s_{i_{t+2}} \cdots s_{i_{r-1}}(\alpha_{i_r})$ is positive. For any root α and $v \in W$, $\gamma = v(\alpha)$ we have $s_\gamma = v s_\alpha v^{-1}$.

(b) Assume that our expression is reduced. Deduce that $s_{i_1} s_{i_2} \cdots s_{i_r}(\alpha_{i_r})$ is negative.

(c) Assume that our expression is reduced. Define β_j for $1 \leq j \leq r$ by

$$\begin{aligned} \beta_1 &= \alpha_{i_1}. \\ \beta_2 &= s_{i_1}(\alpha_{i_2}), \\ \beta_2 &= s_{i_1} s_{i_2}(\alpha_{i_3}), \\ \ldots \\ \beta_r &= s_{i_1} s_{i_2} \ldots s_{i_{r-1}}(\alpha_{i_r}). \end{aligned}$$

Show: The β_j's form exactly the set $S(w^{-1})$ of positive roots that are mapped to negative roots under w^{-1}.

Problem 35. Consider the Weyl group $W = S_n$ and the subgroup $H := S_i \times S_{n-i}$ for $0 \leq i \leq n$.

Show that there is exactly one coset representative of minimal length for each coset $H \backslash W$; give explicitly reduced expressions for these representatives.

Hint: Start to make a complete list for small examples.

Problem 36.

(a) Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $T(\mathfrak{g})$ its tensor algebra and $U(\mathfrak{g})$ its enveloping algebra. The standard filtration $T(\mathfrak{g})_n = \bigoplus_{j \leq n} \mathfrak{g}^{\otimes j}$ of $T(\mathfrak{g})$ induces a filtration on the quotient $U(\mathfrak{g})$. Show that the associated graded algebra is isomorphic to the symmetric algebra $S(\mathfrak{g})$:

$$S(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr} U(\mathfrak{g}).$$

(b) Show that $U(\mathfrak{g})$ is a Noetherian domain.

Now let $U_q(\mathfrak{g})$ be a quantized enveloping algebra (over a field of characteristic zero, q generic). We use the notation of the lecture.

Verify the following general results for $U_q(\mathfrak{sl}_3)$:

(c) (Levendorskii-Soibelman) For $1 \leq i < j \leq N$ we have

$$E_{\beta_i} E_{\beta_j} - q^{(\beta_i, \beta_j)} E_{\beta_j} E_{\beta_i} = \sum_{k \in \mathbb{N}^N} z_k E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N},$$

where $z_k \in \mathbb{Q}[q^{\pm 1}]$ and $z_k = 0$ unless $k_r = 0$ for $r \leq i$ and $r \geq j$.

(d) Consider the filtration of $U_q(\mathfrak{g})$ from the lecture. The associated graded algebra $\operatorname{gr} U_q(\mathfrak{g})$ is isomorphic to the k-algebra with generators $E_{\beta_1}, \dots, E_{\beta_N}$, $F_{\beta_1}, \dots, F_{\beta_N}$, K_α ($\alpha \in Q$), subject to the relations

$$\begin{aligned}
K_\alpha K_\beta &= K_{\alpha+\beta}, & K_0 &= 1 \\
K_\alpha E_{\beta_i} &= q^{(\alpha, \beta_i)} E_{\beta_i} K_\alpha, & K_\alpha F_{\beta_i} &= q^{-(\alpha, \beta_i)} F_{\beta_i} K_\alpha, \\
E_{\beta_i} F_{\beta_j} &= F_{\beta_j} E_{\beta_i}, & & \\
E_{\beta_i} E_{\beta_j} &= q^{(\beta_i, \beta_j)} E_{\beta_j} E_{\beta_i}, & F_{\beta_i} F_{\beta_j} &= q^{(\beta_i, \beta_j)} F_{\beta_j} F_{\beta_i},
\end{aligned}$$

(One can also deduce that $U_q(\mathfrak{g})$ is a Noetherian domain.)