Quantum Groups
Winter Semester 2008/09

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Exercise sheet 1

Solutions to be handed in on Wednesday 22nd October 2008

Problem 1. Let q and r be indeterminates.

(a) Verify the formula

$$\begin{bmatrix} m+1 \\ n \end{bmatrix}_q = q^{-n} \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{m+1-n} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$$

(b) Consider $k[r^{\pm 1}]\langle X, Y\rangle / \langle YX = r^2 XY \rangle$ and show

$$(X+Y)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_r r^{(n-i)i} X^i Y^{n-i}$$

(c) Consider the formal power series in z

$$e(z) = \sum_{n \in \mathbb{N}} \frac{z^n}{[n]!_r \; r^{n(n-1)/2}}$$

with coefficients in $k(r)$. If $yx = r^2 xy$ show that

$$e(x+y) = e(x)e(y).$$

Problem 2. Let k be a field and $q \in k \setminus \{0\}$. Consider the k-algebra $\mathcal{O}_q(M_2(k))$, given by generators a, b, c, d and relations

$$\begin{aligned} ba &= q^{-1}ab & ca &= q^{-1}ac & db &= q^{-1}bd \\ dc &= q^{-1}cd & cb &= bc & da &= ad - (q-q^{-1})bc. \end{aligned}$$

Show that $\{a^i b^j c^l d^m \mid i, j, l, m \in \mathbb{N}\}$ is a basis of $\mathcal{O}_q(M_2(k))$.

Hint: You can use the diamond lemma (without proof): Let $X = \{a, b, c, \dots\}$ be a finite set of letters, $F = k\langle X \rangle$ the free algebra on X and W the free monoid on X (the words in X). Let

$$S = \{(w_\sigma, f_\sigma) \mid \sigma \in \Sigma\} \subset W \times F$$

be a subset.

Given $\sigma \in \Sigma$ and u, $v \in W$, the k-linear map $F \to F$ sending $uw_\sigma v$ to $uf_\sigma v$ and fixing all other words is called an **elementary reduction**. A **reduction** is a finite composition of elementary reductions. An element of F is **irreducible** if it is fixed by all reductions.

Let $\leq$ be the length-lexicographic ordering on W (i. e. $v \leq w$ if either length$(v) <$ length(w) or (length$(v) =$ length(w) and $v \leq_{\text{lex}} w$ in the lexicographic ordering)). This ordering $\leq$ is **compatible** with S if for alle $\sigma \in \Sigma$, f_σ is a linear combination of words $w < w_\sigma$.

An **overlap ambiguity** is is a 5-tuple $(t, u, v, \sigma, \tau) \in W^3 \times \Sigma^2$ such that $tu = w_\sigma$ and $uv = w_\tau$; it is **resolvable** if there are reductions r, r' such that $r(f_\sigma v) = r'(tf_\tau)$.

An **inclusion ambiguity** is is a 5-tuple $(t, u, v, \sigma, \tau) \in W^3 \times \Sigma^2$ such that $tuv = w_\sigma$ and $u = w_\tau$; it is **resolvable** if there are reductions r, r' such that $r(f_\sigma) = r'(tf_\tau v)$.

Diamond Lemma: If $\leq$ is compatible with S and all overlap and inclusion ambiguities are resolvable then the cosets $\overline{w}$ of the irreducible words $w \in W$ form a basis of the factor algebra $F/\langle w_\sigma - f_\sigma \mid \sigma \in \Sigma \rangle$.