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Categorification of Wedderburn's basis for $\mathbb{C}[S_n]$

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Abstract. M. Neunhöffer studies in [21] a certain basis of $\mathbb{C}[S_n]$ with the origins in [14] and shows that this basis is in fact Wedderburn's basis, hence decomposes the right regular representation of S_n into a direct sum of irreducible representations (i.e. Specht or cell modules). In the present paper we rediscover essentially the same basis with a categorical origin coming from projective-injective modules in certain subcategories of the BGG-category $\mathcal{O}$. Inside each of these categories, there is a dominant projective module which plays a crucial role in our arguments and will additionally be used to show that *Kostant's problem* ([10]) has a negative answer for some simple highest weight module over the Lie algebra $\mathfrak{sl}_4$. This disproves the general belief that Kostant's problem should have a positive answer for all simple highest weight modules in type A.

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1. The main result. Let n be a positive integer and S_n the group of permutations of the elements from $\{1, 2, \dots, n\}$. Denote by $\mathcal{S}$ the usual set of Coxeter generators of S_n and by $\mathcal{H} = \mathcal{H}(S_n, \mathcal{S})$ the associated (generic) Iwahori-Hecke algebra. The algebra $\mathcal{H}$ is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_w | w \in S_n\}$ and multiplication given by

$$H_x H_y = H_{xy} \text{ if } l(x) + l(y) = l(xy) \text{ and } H_s^2 = H_e + (v^{-1} - v)H_s \text{ for } s \in S,$$

where $l : S_n \to \mathbb{Z}$ denotes the length function with respect to $\mathcal{S}$. Denote by $\{\underline{H}_w | w \in S_n\}$ the *Kazhdan-Lusztig basis* (in the normalization of [25]). We

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also denote by $\{\underline{\hat{H}}_w | w \in S_n\}$ the dual Kazhdan-Lusztig basis of $\mathcal{H}$, defined via $\tau(\underline{\hat{H}}_v \underline{H}_{w^{-1}}) = \delta_{v,w}$, where τ is the standard symmetrizing trace form.

The group algebra $\mathbb{C}[S_n]$ of S_n is obtained by specializing v to 1 in $\mathcal{H}$, more precisely: by extending first the scalars in $\mathcal{H}$ to $\mathbb{C}$ and then factoring out the ideal generated by $v-1$ we get an epimorphism of $\mathbb{C}$-algebras, which we call the *evaluation map*:

$$\mathrm{ev} : \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H} \overset{\mathrm{proj}}{\twoheadrightarrow} (\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}) / (v-1) \xrightarrow{\sim} \mathbb{C}[S_n], \quad 1 \otimes H_w \mapsto w.$$

The Robinson-Schensted correspondence (see e.g. [24, 3.1]) defines a bijection between elements $w \in S_n$ and pairs $(a(w), b(w))$ of standard tableaux with n boxes, such that $a(w)$ and $b(w)$ are of the same shape. For every element $w \in S_n$ we denote by $\mathtt{R}_w = \{x \in S_n \mid a(x) = a(w)\}$ the *right cell* of S_n which contains w. Let $\overline{w}$ denote the unique involution in $\mathtt{R}_w$. Beside $a(\overline{w}) = a(w)$ the element $\overline{w}$ satisfies (and is characterized by the property) $a(\overline{w}) = b(\overline{w})$. It is the *Duflo involution* of $\mathtt{R}_w$.

Our main result is the construction of a basis $\{f_w | w \in S_n\}$ of $\mathbb{C}[S_n]$ compatible with its regular right S_n-module structure in the following way:

Theorem 1. *For $w \in S_n$ set $f_w = \mathrm{ev}(\underline{\hat{H}}_{\overline{w}} \underline{H}_w)$. Then the following holds:*

(a) *The elements $\{f_w | w \in S_n\}$ form a basis of $\mathbb{C}[S_n]$.*
(b) *Let $x \in S_n$ and consider the linear span $\mathbf{S}(x)$ of all f_w, $w \in \mathtt{R}_x$. Then $\mathbf{S}(x)$ is invariant with respect to the right action of $\mathbb{C}[S_n]$ and isomorphic to the (irreducible) cell module associated with $\mathtt{R}_x$.*

In other words, there is a decomposition of the right regular representation of S_n into a direct sum of irreducible modules which is compatible with the basis $\{f_w | w \in S_n\}$. In fact the theorem and its proof are valid over any field of characteristic zero. As an example, for $n = 3$ let s and t be the simple reflections, then our basis consists of the elements

$$\begin{aligned}
f_e &= (e - s - t + st + ts - sts)e = e - s - t + st + ts - sts, \\
f_s &= (s - ts - st + sts)(s + e) = e + s - t - ts, \\
f_t &= (t - ts - st + sts)(t + e) = e + t - s - st, \\
f_{st} &= (s - ts - st + sts)(st + s + t + e) = s + st - ts - sts, \\
f_{ts} &= (t - ts - st + sts)(ts + s + t + e) = t + ts - st - sts, \\
f_{sts} &= sts(e + t + s + st + ts + sts) = e + t + s + st + ts + sts.
\end{aligned}$$

Unfortunately, this method does not give a basis for the algebra $\mathcal{H}$.

Theorem 1 turns out to be related to the paper [21], where a similar basis was studied. Let $\{\mathtt{R}_i : i \in I\}$ be a set of right cells in S_n containing exactly one representative of each two-sided sell. For each $i \in I$ and $(x, y) \in \mathtt{R}_i \times \mathtt{R}_i$ set $h^i_{(x,y)} = \mathrm{ev}(\underline{\hat{H}}_{x^{-1}} \underline{H}_y)$. From [21] it follows that $\{h^i_{(x,y)} | i \in I, (x, y) \in \mathtt{R}_i \times \mathtt{R}_i\}$ has properties analogous to those of the basis $\{f_w | w \in S_n\}$ from Theorem 1. Moreover, in [21] it is even proved that a normalized version of $\{h^i_{(x,y)} | i \in I, (x,y) \in \mathtt{R}_i \times \mathtt{R}_i\}$ is in fact Wedderburn's basis of $\mathbb{C}[S_n]$ (i.e. basis elements correspond to matrix units in Wedderburn's decomposition of $\mathbb{C}[S_n]$). The origins of the basis $\{h^i_{(x,y)} | i \in I, (x,y) \in \mathtt{R}_i \times \mathtt{R}_i\}$ go further back to [14]. There is an asymptotic version $\mathcal{J}$ of the Hecke algebra, introduced by Lusztig in [14] together with a homomorphism $\Psi : \mathcal{H} \to \mathbb{Z}[v, v^{-1}] \otimes_{\mathbb{Z}} \mathcal{J}$ which becomes an isomorphism over $\mathbb{Q}(t)$. As pointed out to us by Neunhöffer, the basis $\{h^i_{(x,y)} | i \in I, (x,y) \in \mathtt{R}_i \times \mathtt{R}_i\}$ is exactly Lusztig's basis for $\mathcal{J}$ pulled back via the homomorphism Ψ to $\mathcal{H}$. The connection to [21] is the following:

Theorem 2. $\{f_w | w \in S_n\} = \{h^i_{(x,y)} | i \in I, (x,y) \in \mathtt{R}_i \times \mathtt{R}_i\}$.

The origins of Theorem 1, as well as the proof of Theorem 2, are categorical; and this is absolutely crucial for our arguments. In particular, our setup is completely different from the combinatorial approach of [21]. There are alternative combinatorial approaches to the construction of a basis for $\mathbb{C}[S_n]$ and some related algebras in which the regular representation decomposes into a direct sum of irreducibles, see [22], [19], [20], [15], [16]. There are also alternative combinatorial constructions (e.g. [11], [1], [2]) giving decompositions of the regular representation of S_n into irreducible representations using an explicit basis, which lead only to filtrations whose successive subquotients are irreducible.

2. Proof of Theorem 1. We prove Theorem 1 by giving an explicit categorical interpretation of all ingredients, which is based on the categorification of cell modules as established in [18, Section 4] (the original idea of categorifying the Hecke algebra goes back to [11] and [4]). The main players here are certain subquotient categories of the famous BGG category $\mathcal{O}$ (for the latter see [5]).

Let $\mathcal{O}_0$ be the principal block of $\mathcal{O}$ for the simple complex Lie algebra $\mathfrak{sl}_n$ with its standard triangular decomposition. The simple objects in $\mathcal{O}_0$ are the $L(w)$, $w \in S_n$, the simple highest weight modules with the highest weight $w(\rho) - \rho$, where ρ is the half-sum of all positive roots. Let $\Delta(w)$ and $P(w)$ denote the Verma and the indecomposable projective module with unique simple quotient isomorphic to $L(w)$ respectively. Further, denote by θ_w the indecomposable projective endofunctor of $\mathcal{O}_0$ with the property $\theta_w P(e) \cong \theta_w \Delta(e) \cong P(w)$ (see [4]). Finally, let $[\mathcal{O}_0]$ denote the complexified Grothendieck group of $\mathcal{O}_0$. For $M \in \mathcal{O}_0$ we denote by $[M]$ its image in $[\mathcal{O}_0]$.

There is a $\mathbb{C}$-linear isomorphism $\varphi : [\mathcal{O}_0] \to \mathbb{C}[S_n]$ with $\varphi([\Delta(w)]) = w$. The Kazhdan-Lusztig conjecture ([11], proved in [3], [7]) implies that $\varphi([P(w)]) = \mathrm{ev}(\underline{H}_w)$ (for an overview see e.g. [18, Subsection 3.4]). The standard bilinear form on $\mathbb{C}[S_n]$ is categorified via the bifunctor $\mathrm{Ext}^*(\_, \_)$ ([13, Section 5] or [18, Subsection 4.6]). Indecomposable projective and simple modules form dual bases with respect to this form, and hence

$$\varphi([L(w)]) = \mathrm{ev}(\underline{\hat{H}}_w). \tag{2.1}$$

The functors θ_w are exact and induce therefore $\mathbb{C}$-linear endomorphisms $[\theta_w]$ of $[\mathcal{O}_0]$. By [4, Theorem 3.4(iv)] and [25] (for a more adjusted reformulation see [18, Subsection 3.4]) we have

$$\varphi([\theta_w M]) = \varphi([\theta_w][M]) = \varphi([M])\mathrm{ev}(\underline{H}_w). \tag{2.2}$$

for all M in $\mathcal{O}_0$. Recall the right cells mentioned above and let $\leq_{\mathtt{R}}$ be the right preorder on S_n. Fix $w \in W$ and set $\hat{\mathtt{R}}_w = \{x \in S_n | x \leq_{\mathtt{R}} y \text{ for some } y \in \mathtt{R}_w\}$. Associated with the right cell $\mathtt{R}_w$ of w we have the full subcategory $\mathcal{O}_0^{\hat{\mathtt{R}}_w}$ of $\mathcal{O}_0$, which consists of all modules M with all composition subquotients of the form $L(x)$ with $x \in \hat{\mathtt{R}}_w$. Let $\mathrm{Z}^{\hat{\mathtt{R}}_w} : \mathcal{O}_0 \to \mathcal{O}_0^{\hat{\mathtt{R}}_w}$ be the natural projection functor which takes the maximal quotient that lies in $\mathcal{O}_0^{\hat{\mathtt{R}}_w}$. All this is built up such that we have

$$\mathrm{Z}^{\hat{\mathtt{R}}_w}\theta_x \cong \theta_x \mathrm{Z}^{\hat{\mathtt{R}}_w} \tag{2.3}$$

for any $x, w \in S_n$, ([18, Lemma 19]). For $x \in S_n$ we define $P^{\hat{\mathtt{R}}_w}(x) = \mathrm{Z}^{\hat{\mathtt{R}}_w} P(x)$, and it follows that

$$P^{\hat{\mathtt{R}}_w}(x) \neq 0 \text{ if and only if } x \in \hat{\mathtt{R}}_w. \tag{2.4}$$

Moreover, the set $\{P^{\hat{\mathtt{R}}_w}(x) | x \in \hat{\mathtt{R}}_w\}$ constitutes a complete list of indecomposable projective modules in $\mathcal{O}_0^{\hat{\mathtt{R}}_w}$.

The following provides a basis of $\mathbb{C}[S_n]$ with most of the desired properties:

Proposition 3. *For $w \in S_n$ define $g_w = \varphi([P^{\hat{\mathtt{R}}_w}(w)]) \in \mathbb{C}[S_n]$. Then the following holds:*

(a) *$\{g_w | w \in S_n\}$ is a basis of $\mathbb{C}[S_n]$.*
(b) *For every $x \in S_n$ the linear span of $\{g_w | w \in \mathtt{R}_x\}$ is invariant with respect to the right action of S_n and is isomorphic to the cell module associated with $\mathtt{R}_x$.*

Proof. As $|\{g_w | w \in S_n\}| = |S_n| = \dim_{\mathbb{C}} \mathbb{C}[S_n]$, it is enough to show that the elements from $\{g_w | w \in S_n\}$ are linearly independent. By definition of the category $\mathcal{O}_0^{\hat{\mathtt{R}}_x}$, all the simple composition factors of $P^{\hat{\mathtt{R}}_x}(w)$ are of the form $L(z)$ where z is smaller or equal to x in the right cell order. Therefore, when expressed in the specialization $\{\mathrm{ev}(\underline{\hat{H}}_z) \mid z \in S_n\}$ of the dual Kazhdan-Lusztig basis, the element g_w is a linear combination of basis elements, corresponding to $z \in \hat{\mathtt{R}}_x$ (see (2.1)). By induction on the right order, it is then enough to show that for any $x \in S_n$ the elements from $\{g_w | w \in \mathtt{R}_x\}$ are linearly independent. By [13, Theorem 1] and [18, Theorem 18], these elements form the Kazhdan-Lusztig basis in the cell module associated with $\mathtt{R}_x$. The cell module is a subquotient of $\mathbb{C}[S_n]$. Hence these elements are linearly independent already in $\mathbb{C}[S_n]$. The first statement follows.

To prove the invariance it is enough to show, thanks to (2.2), that projective functors preserve the additive subcategory $\mathscr{A}$ of $\mathcal{O}_0^{\hat{\mathtt{R}}_x}$ generated by the indecomposable projective modules $P^{\hat{\mathtt{R}}_x}(w)$, $w \in \mathtt{R}_x$. Since $\mathcal{H}$ is generated by the $\underline{H}_s$, where s runs through $\mathcal{S}$, it is enough to show that for any $s \in \mathcal{S}$ and $w \in \mathtt{R}_x$ the module

$\theta_s P^{\hat{\mathtt{R}}_x}(w)$ belongs to $\mathscr{A}$. Now (2.3), [18, (4.1)] and (2.4) provide the following three isomorphisms:

$$\begin{aligned}\theta_s P^{\hat{\mathtt{R}}_x}(w) = \theta_s \mathrm{Z}^{\hat{\mathtt{R}}_x}\theta_w\Delta(e) &\cong \mathrm{Z}^{\hat{\mathtt{R}}_x}\theta_s\theta_w\Delta(e)\\ &\cong \mathrm{Z}^{\hat{\mathtt{R}}_x}\left(\oplus_{y\geq_{\mathtt{R}} w}\theta_y^{m_y}\Delta(e)\right) = \oplus_{y\geq_{\mathtt{R}}\mathtt{R}_x}(\mathrm{Z}^{\hat{\mathtt{R}}_x}P(y))^{\oplus m_y}\\ &\cong \oplus_{y\in\mathtt{R}_x}\oplus_{i=1}^{m_y} P^{\hat{\mathtt{R}}_w}(y)\end{aligned}$$

for some non-negative integers m_y. The claim about the invariance follows. The claim about the cell module follows from [18, Theorem 16 and Theorem 18]. □

Now Theorem 1 follows from the following statement:

Proposition 4. *We have $f_w = g_w$ for all $w \in S_n$. In particular, Theorem 1 follows from Proposition 3.*

Proof. We already know that $\varphi([L(w)]) = \mathrm{ev}(\underline{\hat{H}}_w)$ for all $w \in S_n$. Thanks to (2.2) and the definitions of f_w and g_w, the proposition is implied by the

Key statement: Let $w \in S_n$, then $\theta_w L(\overline{w}) \cong P^{\hat{\mathtt{R}}_w}(w)$,

which also explains the categorical meaning of the basis. In what follows we prove this statement.

Recall that $P^{\hat{\mathtt{R}}_w}(w) \cong \theta_w P^{\hat{\mathtt{R}}_w}(e)$ by (2.3). To prove the key statement we have to study the dominant projective module $P^{\hat{\mathtt{R}}_w}(e)$ in $\mathcal{O}_0^{\hat{\mathtt{R}}_w}$ in more detail.

Lemma 5. *Let $x \in \mathtt{R}_w$ be such that $x \neq \overline{w}$. Then $[P^{\hat{\mathtt{R}}_w}(e) : L(x)] = 0$.*

Proof. Recall that the functor θ_x is both left and right adjoint to the functor $\theta_{x^{-1}}$. Hence we have

$$\begin{aligned}[P^{\hat{\mathtt{R}}_w}(e) : L(x)] &= \dim \mathrm{Hom}_{\mathcal{O}}(P^{\hat{\mathtt{R}}_w}(x), P^{\hat{\mathtt{R}}_w}(e))\\ &= \dim \mathrm{Hom}_{\mathcal{O}}(\theta_x P^{\hat{\mathtt{R}}_w}(e), P^{\hat{\mathtt{R}}_w}(e))\\ &= \dim \mathrm{Hom}_{\mathcal{O}}(P^{\hat{\mathtt{R}}_w}(e), \theta_{x^{-1}} P^{\hat{\mathtt{R}}_w}(e)).\end{aligned}$$

As $x \neq \overline{w}$, we have $x \neq x^{-1}$, and hence, using [24, Theorem 3.6.6], we have $a(x^{-1}) = b(x) \neq a(x)$. Thus $x^{-1} \notin \mathtt{R}_w$. Since $a(x^{-1})$ and $a(x)$ still have the same shape, it follows that $x^{-1} \notin \hat{\mathtt{R}}_w$ ([6, Exercise 10, p. 198]). Therefore $\theta_{x^{-1}} P^{\hat{\mathtt{R}}_w}(e) = \theta_{x^{-1}}\mathrm{Z}^{\hat{\mathtt{R}}_w}\Delta(e) \cong \mathrm{Z}^{\hat{\mathtt{R}}_w}\theta_{x^{-1}}\Delta(e) \cong \mathrm{Z}^{\hat{\mathtt{R}}_w}P(x^{-1}) = 0$ and thus $\dim \mathrm{Hom}_{\mathcal{O}}(P^{\hat{\mathtt{R}}_w}(e), \theta_{x^{-1}} P^{\hat{\mathtt{R}}_w}(e)) = 0$ as well. □

Lemma 6. *For any $x \in \mathtt{R}_w$ and $y \in \hat{\mathtt{R}}_w \setminus \mathtt{R}_w$ we have $\theta_x L(y) = 0$. In particular, $[P^{\hat{\mathtt{R}}_w}(e) : L(\overline{w})] > 0$.*

Proof. As $P^{\hat{\mathtt{R}}_w}(y) \twoheadrightarrow L(y)$ and θ_x is exact, we have $\theta_x P^{\hat{\mathtt{R}}_w}(y) \twoheadrightarrow \theta_x L(y)$. Applying (2.3) we even have that $\theta_x L(y)$ is a homomorphic image of the module $\mathrm{Z}^{\hat{\mathtt{R}}_w}\theta_x\theta_y\Delta(e)$.

Note that $\theta_x L(y) \in \mathcal{O}_0^{\hat{\mathtt{R}}_y}$, in particular, all simple subquotients of $\theta_x L(y)$ have the form $L(z)$, $z \in \hat{\mathtt{R}}_y$.

On the other hand, it follows from [18, (4.1)] that $\theta_x\theta_y$ is a direct sum of functors of the form θ_z, where $z \geq_{\mathtt{L}} x$. Hence, by (2.4), all simple quotients of the module $\mathrm{Z}^{\hat{\mathtt{R}}_w}\theta_x\theta_y\Delta(e)$ have the form $L(x)$. As $x \notin \hat{\mathtt{R}}_y$ by our choice of y, we must have $\theta_x L(y) = 0$.

We know that $P^{\hat{\mathtt{R}}_w}(\overline{w}) = \theta_{\overline{w}} P^{\hat{\mathtt{R}}_w}(e) \neq 0$. By Lemma 5 and the above, $L(\overline{w})$ is the only subquotient of $P^{\hat{\mathtt{R}}_w}(e)$ which has the chance not to be annihilated by $\theta_{\overline{w}}$. Altogether we must have $[P^{\hat{\mathtt{R}}_w}(e) : L(\overline{w})] > 0$ □

Lemma 7. $[P^{\hat{\mathtt{R}}_w}(e) : L(\overline{w})] = 1$.

Proof. Assume for a moment that $\mathtt{R}_w$ contains an element of the form $w_0'w_0$, where w_0 is the longest element of S_n and w_0' is the longest element of some parabolic (Young) subgroup W of S_n. Let S be the set of simple reflections in W. Then the modules $P^{\hat{\mathtt{R}}_w}(x)$, $x \in \mathtt{R}_w$, are exactly the indecomposable projective-injective modules in the parabolic subcategory $\mathcal{O}_0^S$ (in the sense of [23]) of $\mathcal{O}_0$ ([18, Remark 14]). Amongst the indecomposable projective-injective modules in $\mathcal{O}_0^S$ there is, due to [8, 3.1], a special one which is obtained as a translation of some simple projective module (out of possibly several walls). Since translation to walls maps simple modules to simples or zero, the special module, call it P, is thus obtained as a translation of some $L(x)$ for some $x \in \mathtt{R}_w$.

From [13, Theorem 1] it further follows that translating P and taking appropriate direct summands, we will finally get all $P^{\hat{\mathtt{R}}_w}(x)$, $x \in \mathtt{R}_w$. This implies the existence of an indecomposable projective functor θ_y such that the module $\theta_y L(\overline{w})$ contains $P^{\hat{\mathtt{R}}_w}(\overline{w})$ as a direct summand (see [18, 5.1]). By [18, Theorem 18], the above restriction that the right cell should contain $w_0'w_0$ is in fact superfluous. Moreover, from [18, Theorem 18] it also follows that the module $P^{\hat{\mathtt{R}}_w}(\overline{w})$ is an injective object in $\mathcal{O}^{\hat{\mathtt{R}}_w}$ (and so the same holds for any $P^{\hat{\mathtt{R}}_w}(x)$, $x \in \mathtt{R}_w$).

Consider now $\theta_y P^{\hat{\mathtt{R}}_w}(e) \cong P^{\hat{\mathtt{R}}_w}(y)$. As $P^{\hat{\mathtt{R}}_w}(\overline{w})$ is both projective and injective, from Lemma 6 it follows that $P^{\hat{\mathtt{R}}_w}(\overline{w})$ must be a direct summand of $P^{\hat{\mathtt{R}}_w}(y)$. As $P^{\hat{\mathtt{R}}_w}(y)$ is indecomposable, this forces $P^{\hat{\mathtt{R}}_w}(y) \cong P^{\hat{\mathtt{R}}_w}(\overline{w})$, $y = \overline{w}$, and finally $[P^{\hat{\mathtt{R}}_w}(e) : L(\overline{w})] = 1$. □

From Lemma 6 and Lemma 7 it follows that for any $x \in \mathtt{R}_w$ we have $\theta_x P^{\hat{\mathtt{R}}_w}(e) \cong \theta_x L(\overline{w})$. This finally proves the key statement and at the same time completes the proof of Proposition 4 and Theorem 1. □

Remark 8. Let $w \in S_n$ be such that the right cell $\mathtt{R}_w$ contains the element $w_0'w_0$ for some Young subgroup W' of S_n. Then $\mathcal{O}_0^{\hat{\mathtt{R}}_w}$ is the regular block of the parabolic category $\mathcal{O}$ (in the sense of [23]) associated with W'. The elements f_x, $x \not\leq_{\mathtt{R}} w$, form a basis of a submodule N of $\mathbb{C}[S_n]$. The quotient $\mathbb{C}[S_n]/N$ is isomorphic to the induced sign module $\mathbb{C}[S_n] \otimes_{\mathbb{C}[W]}$ sign (see [18, 6.2.1] for details) with the classes of the elements f_x, $x \leq_{\mathtt{R}} w$ forming a basis. Alternatively, the elements f_x, $x \leq_{\mathtt{R}} w$, form a basis of a submodule of $\mathbb{C}[S_n]$ which is isomorphic to the induced sign module.

3. Proof of Theorem 2. Using (2.1) and (2.2) we interpret $h^i_{(x,y)} = \varphi([\theta_y L(x^{-1})])$ for each $i \in I$ and $(x, y) \in \mathtt{R}_i \times \mathtt{R}_i$. Let $i \in I$ be fixed. Because of Proposition 4 and the definition of g_w's, to prove Theorem 2 it is enough to show that every $\theta_y L(x^{-1})$ is a projective-injective module in $\mathcal{O}_0^{\hat{\mathtt{R}}_{x^{-1}}}$. In the case $x = \overline{y}$ this follows from the Key statement of Section 2.

Let now $x \in \mathtt{R}_i$ be arbitrary. As x and $\overline{y}$ belong to the same right cell, the elements x^{-1} and $\overline{y}$ belong to the same left cell. Let $\mathscr{A}$ and $\mathscr{B}$ denote the additive categories of projective-injective modules in $\mathcal{O}_0^{\hat{\mathtt{R}}_{\overline{y}}}$ and $\mathcal{O}_0^{\hat{\mathtt{R}}_{x^{-1}}}$ respectively. In [18, Section 5] it was shown that there exists an equivalence $\mathrm{F} : \mathscr{A} \to \mathscr{B}$ which commutes with projective functors and satisfies $\mathrm{F}(P^{\hat{\mathtt{R}}_{\overline{y}}}(\overline{y})) = P^{\hat{\mathtt{R}}_{x^{-1}}}(x^{-1})$.

Let $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ denote the full subcategories of respectively $\mathcal{O}_0^{\hat{\mathtt{R}}_{\overline{y}}}$ and $\mathcal{O}_0^{\hat{\mathtt{R}}_{x^{-1}}}$ which consist of all modules X having a two step presentation $M_1 \to M_0 \to X \to 0$, where $M_1, M_0 \in \mathscr{A}$ or $M_1, M_0 \in \mathscr{B}$ respectively. Then F extends, in the obvious way, to an equivalence $\overline{\mathrm{F}} : \overline{\mathscr{A}} \to \overline{\mathscr{B}}$ which commutes with projective functors.

Let $\overline{L(\overline{y})}$ denote the quotient of $P^{\hat{\mathtt{R}}_{\overline{y}}}(\overline{y})$ modulo the trace of all modules from $\mathscr{A}$ in the radical of $P^{\hat{\mathtt{R}}_{\overline{y}}}(\overline{y})$. Define $\overline{L(x^{-1})}$ analogously. Then $\overline{L(\overline{y})}$ has simple top $L(\overline{y})$ and all other subquotients of $\overline{L(\overline{y})}$ are of the form $L(z)$, where $z <_{\mathtt{R}} \overline{y}$. Analogously $\overline{L(x^{-1})}$ has simple top $L(x^{-1})$ and all other subquotients of $\overline{L(x^{-1})}$ are of the form $L(z)$, where $z <_{\mathtt{R}} x^{-1}$. From the above construction we have $\overline{\mathrm{F}}(\overline{L(\overline{y})}) = \overline{L(x^{-1})}$. Further $\theta_y \overline{L(\overline{y})} = \theta_y L(\overline{y})$ by Lemma 6. Analogous arguments imply $\theta_y \overline{L(x^{-1})} = \theta_y L(x^{-1})$. Adding everything up we have

$$\theta_y L(x^{-1}) = \theta_y \overline{L(x^{-1})} = \theta_y \overline{\mathrm{F}}(\overline{L(\overline{y})}) = \overline{\mathrm{F}}(\theta_y \overline{L(\overline{y})}) = \overline{\mathrm{F}}(\theta_y L(\overline{y})) = \mathrm{F}(\theta_y L(\overline{y})).$$

Hence $\theta_y L(x^{-1}) = \mathrm{F}(\theta_y L(\overline{y}))$ is a projective-injective module in $\mathcal{O}_0^{\hat{\mathtt{R}}_{x^{-1}}}$. The claim follows.

4. An application to Kostant's problem. The core object $\Delta^{\hat{\mathtt{R}}_w}(e)$ of our study in Section 2 has an unexpected application to the so-called Kostant's problem from [10]; see also [9, Kapitel 6].

Let $\mathfrak{g}$ be a complex reductive finite-dimensional Lie algebra. For every $\mathfrak{g}$-module M we have the bimodule $\mathscr{L}(M, M)$ of all $\mathbb{C}$-linear endomorphisms of M on which the adjoint action of the universal enveloping algebra $U(\mathfrak{g})$ is locally finite. (That means any vector $f \in \mathscr{L}(M,M)$ lies inside a finite dimensional subspace which is stable under the adjoint action defined as $x.f(m) = x(f(m)) - f(xm)$ for $x \in \mathfrak{g}$, $m \in M$). Initiated by [10], *Kostant's problem* became the standard terminology for the following question concerning an arbitrary $\mathfrak{g}$-module M:

Is the natural injection $U(\mathfrak{g})/\mathrm{Ann}(M) \hookrightarrow \mathscr{L}(M,M)$ *surjective?*

Although there are several classes of modules for which the answer is known to be positive (see [10], [17], [18] and references therein), a complete answer to this problem seems to be far away - the problem is not even solved for simple highest weight modules. In [10, 9.5] an example of a simple highest weight module in type B_2, for which the answer is negative is mentioned (for details see [18, 11.5]). In this section we use the module $\Delta^{\hat{\mathtt{R}}_x w}(e)$ to construct another example in type A_3, which disproves a general belief that the answer to Kostant's problem is positive for simple highest weight modules in type A (this belief was based on [10, 9.1] and further strengthened by [18, Theorem 60]).

Let $n = 4$ and $r = (12)$, $s = (23)$, $t = (34)$ be the standard Coxeter generators of S_4. Consider $w = rt = \overline{w}$. In this case we have $\mathtt{R}_w = \{rt, rts\}$ and $\hat{\mathtt{R}}_w = \{rt, rts, t, ts, tsr, r, rs, rst, e\}$. We consider the graded version of $\mathcal{O}$ as worked out in [26]. Using [27, Appendix] one computes that the module $N = \Delta^{\hat{\mathtt{R}}_w}(e)$ has the following graded filtration (resp. socle or radical filtration), where we abbreviate $L(x)$ simply by x:

$$N = \begin{array}{ccc} & e & \\ r & & t \\ & rt & \end{array}$$

Lemma 9. $\mathrm{Ann}(L(rt)) = \mathrm{Ann}(N)$

Proof. Let Y_r and Y_t denote some non-zero elements from the negative root spaces corresponding to r and t respectively. Let further U' be the localization of $U(\mathfrak{sl}_4)$ with respect to the multiplicative set $\{Y_r^i Y_t^j | i, j \geq 0\}$. As $rt > r$ and $rt > t$ with respect to the Bruhat order, both Y_r and Y_t act injectively on $L(rt)$. Hence $L(rt)$ will be the simple socle of the $\mathfrak{sl}_4$-module $N' = U' \otimes_{U(\mathfrak{sl}_4)} L(rt)$. As $t > e$ it is further easy to see (for example using the results of [12, Section 4]) that N is a submodule of N'. Hence the statement of the lemma would follow if we would prove that $\mathrm{Ann}(L(rt)) = \mathrm{Ann}(N')$. In fact, as $L(rt) \subset N'$, we have only to prove that $\mathrm{Ann}(L(rt)) \subset \mathrm{Ann}(N')$. This however, follows from the following statement:

Lemma 10. *Let* $\mathfrak{g}$ *be a semi-simple finite-dimensional Lie algebra,* $0 \neq x \in \mathfrak{g}$ *some root vector, and* M *a* $\mathfrak{g}$*-module on which* x *acts injectively. Let* U' *be the localization of* $U(\mathfrak{g})$ *with respect to the powers of* X. *Then* $\mathrm{Ann}(M) \subset \mathrm{Ann}(M')$, *where* $M' = U' \otimes_{U(\mathfrak{g})} M$.

Proof. The set $X := \{x^i \mid i \geq 0\}$ is an Ore set in $\mathcal{U}(\mathfrak{g})$ with $X \cap \mathrm{Ann}(M) = \emptyset$ by hypothesis. So $U'\mathrm{Ann}(M) = \mathrm{Ann}(M)U'$ is a proper ideal in U'. This means $\mathrm{Ann}(M)M' = \mathrm{Ann}(M)U'M = U'\mathrm{Ann}(M)M = \{0\}$. This completes the proof. □

The proof of Lemma 9 is now complete. □

Lemma 11. (a) *The module $\theta_t\theta_s\theta_r N$ has the following graded filtration:*

$$\begin{array}{cccc} & rst & & \\ rs & & rt & \\ rst & tsr & trs & r \\ & & rt & \end{array}$$

(b) *The module $\theta_t\theta_s\theta_r L(rt)$ is a submodule of the module $\theta_t\theta_s\theta_r N$ and has the following graded filtration:*

$$\begin{array}{ccc} & rt & \\ tsr & trs & r \\ & rt & \end{array}$$

Proof. This is verified by direct computations. □

Theorem 12. *Kostant's problem has a negative answer for $L(rt)$.*

Proof. As N is a quotient of the dominant Verma module, Kostant's problem has a positive solution for N by [9, 6.9]. Hence $\mathscr{L}(N,N) = U(\mathfrak{sl}_4)/\mathrm{Ann}(N)$. By Lemma 9, we have $\mathrm{Ann}(N) = \mathrm{Ann}(L(rt))$ and hence we also have $U(\mathfrak{sl}_4)/\mathrm{Ann}(N) = U(\mathfrak{sl}_4)/\mathrm{Ann}(L(rt))$. From Lemma 11(a) we obtain that $\dim \mathrm{Hom}_{\mathcal{O}}(N, \theta_t\theta_s\theta_r N) = 0$ (as for the top $L(e)$ of N we have $[\theta_t\theta_s\theta_r N : L(e)] = 0$), while $\dim \mathrm{Hom}_{\mathcal{O}}(L(rt), \theta_t\theta_s\theta_r L(rt)) \neq 0$ by Lemma 11(b) (as $L(rt)$ obviously occurs in the socle of $\theta_t\theta_s\theta_r L(rt)$). This implies $\mathscr{L}(N,N) \neq \mathscr{L}(L(rt), L(rt))$, which, in turn, yields $\mathscr{L}(L(rt), L(rt)) \neq U(\mathfrak{sl}_4)/\mathrm{Ann}(L(rt))$. The claim follows. □

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