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TWISTING FUNCTORS ON $\mathcal{O}$

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Abstract. This paper studies twisting functors on the main block of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ and describes what happens to (dual) Verma modules. We consider properties of the right adjoint functors and show that they induce an auto-equivalence of derived categories. This allows us to give a very precise description of twisted simple objects. We explain how these results give a reformulation of the Kazhdan-Lusztig conjectures in terms of twisting functors.

Introduction

In the following we study the structure of certain modules for a semisimple complex Lie algebra arising from twisting functors and explain connections to multiplicity formulas for composition factors.

We fix a semisimple complex Lie algebra $\mathfrak{g}$ and choose a Borel and a Cartan subalgebra inside $\mathfrak{g}$. We consider the corresponding BGG-category $\mathcal{O}$ of $\mathfrak{g}$-modules with certain finiteness conditions ([BGG76]). For any element w of the corresponding Weyl group we define, following [Ark], an endofunctor T_w of $\mathcal{O}$ which is given by tensoring with a certain $\mathfrak{g}$-bimodule S_w (a semi-infinite analog of the universal enveloping algebra of $\mathfrak{g}$). Such functors can be defined in a very general setup. In [Soe98], for example, they were used to get character formulas for tilting modules for Kac-Moody algebras. Moreover, they also exist in the quantized situation (see [And03]) and our results should carry over to that case. But in this paper we restrict ourselves to describing their properties in the ordinary category $\mathcal{O}$.

In [AL02] it is shown what happens to a Verma module under these twisting functors. Following [FF90], the resulting modules are called twisted Verma modules since their characters are again characters of Verma modules. They turn out to be exactly the principal series representations which play a crucial role in representation theory ([Jos94], [Irv93]). They can also be realized as local cohomology groups of line bundles on the corresponding flag variety (see [AL02]) and are exactly the Wakimoto modules, i.e., modules with certain vanishing properties for the semi-infinite cohomology (see [Vor99] and [FF90] for the affine case).

Although, there are several different constructions of twisted Verma modules, their "intrinsic" structure (like socle and radical filtrations) is in general unclear. For dual Verma modules, however, the situation is easier, since twisting functors stabilize the set of dual Verma modules (Theorem 2.3).

As a first general result, we deduce (Corollary 4.2) that each twisting functor induces an auto-equivalence on the bounded derived category (as stated by Arkhipov

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in [Ark]). This result is a very strong tool for computing homomorphism spaces and extensions between (twisted) objects. In particular, we can use it to show that any indecomposable module with Verma flag stays indecomposable under twisting. For instance, all twisted Verma modules are indecomposable, and so are the twists of indecomposable projective objects. Since the functors satisfy braid relations we can consider them as a functorial braid group action via auto-equivalences (in the sense of [KS02]).

If s is a simple reflection, the composite T_sT_s is far from being the identity. However, the alternating sum $\Sigma(-1)^i\mathcal{L}_iT_s$ defines an involution on the Grothendieck group of $\mathcal{O}$. Using the right adjoint functor of T_s we get a description of all simple objects annihilated by T_s (Theorem 5.1). Moreover, the trivial simple module L has the following kind of "Borel-Weil-Bott" vanishing property (Corollary 6.2)

$$\mathcal{L}_iT_wL \cong \begin{cases} L & \text{if } i = l(w), \\ 0 & \text{if } i \neq l(w). \end{cases}$$

The interesting case, however, are the nonzero twisted simple modules. Theorem 6.3 gives a very explicit description of such twisted simple objects. These modules are important for understanding the whole category; they cover all the categorical information of $\mathcal{O}$: We get as the main application of the paper a reformulation of the Kazhdan-Lusztig conjecture ([KL79]) in terms of twisted simple objects. At first glance, this statement might not show the significance and impact which it really has, namely:

Claim. For all $x \in W$ not maximal, there exists a simple reflection s such that $L(x.0)$ is a submodule of $T_sL(sx.0)$.

A proof of this claim would imply a proof of the Kazhdan-Lusztig conjecture.

1. Notation and preliminaries

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with a chosen Borel and a fixed Cartan subalgebra. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Cartan decomposition. We denote by $\mathcal{U}(L)$ the universal enveloping algebra for any complex Lie algebra L. Let M be a $\mathcal{U}(\mathfrak{g})$-module. For $\lambda \in \mathfrak{h}^*$ we denote by $M_\lambda = \{m \in M \mid h.m = \lambda(h)m, \forall h \in \mathfrak{h}\}$ the λ-weight space of M. If $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$, then M is called $\mathfrak{h}$-diagonalizable.

Let $R^+ \subset R$ denote the subset of positive roots inside the set of all roots. If we consider $\mathfrak{g}$ as a $\mathcal{U}(\mathfrak{g})$-module via the adjoint action we get by definition $\mathfrak{n}_- = \bigoplus_{\alpha\in R_+} \mathfrak{g}_{-\alpha}$ and $\mathfrak{n} = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$. We fix a Chevalley basis $\{x_\alpha, \mid \alpha \in R\} \cup \{h_i \mid 1 \leq i \leq \operatorname{rank}\mathfrak{g}\}$ of $\mathfrak{g}$. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the corresponding Chevalley anti-automorphism. The positive roots define a partial ordering on $\mathfrak{h}^*$ by letting $\lambda \geq \mu$ if and only if $\lambda - \mu \in \mathbb{N}R^+$. Let W be the Weyl group corresponding to R with a set of simple reflections S generating W and length function l. We denote the shortest element in W by e and the longest by w_o. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots. The *dot*-action of W on $\mathfrak{h}^*$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where the action on the right-hand side is the usual action of the Weyl group on $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, we denote by $W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}$ the stabilizer of λ under the dot-action. If $W_\lambda = \{e\}$, then λ is called *regular*. A weight $\lambda \in \mathfrak{h}^*$ is called *integral* if $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ for any simple root α. Let $\mathfrak{h}^*_{\text{dom}} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda + \rho, \check{\alpha} \rangle \geq 0 \text{ for any simple root } \alpha\}$ be the set of *dominant* weights.

We consider the category $\mathcal{O}$ from [BGG76], i.e., the full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules, where the objects are precisely the modules which are

(1) finitely generated,
(2) $\mathfrak{h}$-diagonalizable, and
(3) $\mathcal{U}(\mathfrak{n})$-locally finite, i.e., $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{n})m < \infty$ for all $m \in M$.

For $\lambda \in \mathfrak{h}^*$ we denote by $\mathbb{C}_\lambda$ the corresponding one-dimensional $\mathfrak{h}$-module considered as a $\mathfrak{b}$-module with trivial $\mathfrak{n}$-action. Let $\Delta(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ denote the *Verma module* with highest weight λ; it is an object in $\mathcal{O}$, its unique simple quotient is denoted by $L(\lambda)$. The modules $L(\lambda)$, where $\lambda \in \mathfrak{h}^*$, are pairwise non-isomorphic and constitute a full set of isomorphism classes of simple modules in $\mathcal{O}$. We denote by $P(\lambda)$ and $I(\lambda)$ the projective cover and injective hull respectively of $L(\lambda)$.

Let M be a $\mathcal{U}(\mathfrak{g})$-module which is $\mathfrak{h}$-diagonalizable. We denote by $\mathrm{d}(M)$ the largest submodule of $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ which is $\mathfrak{h}$-diagonalizable, i.e., $\mathrm{d}(M) = \bigoplus_{\lambda \in \mathfrak{h}^*} (M_\lambda)^*$. The module $\mathrm{d}(M)$ becomes an object in $\mathcal{O}$ with $\mathfrak{g}$-action given by $x.f(m) = f(\sigma(x)m)$ for any $x \in \mathfrak{g}$, $m \in M$ and $f \in \mathrm{d}(M)$. Moreover, d defines a duality on $\mathcal{O}$ fixing simple modules and preserving characters. We let $\nabla(\lambda) = \mathrm{d}\,\Delta(\lambda)$ denote the dual Verma module with submodule $L(\lambda)$. (For details see [BGG76], [Jan79], [Jan83])

In the following tensor products and dimensions are all meant to be (as vector spaces) over the complex numbers if not otherwise stated.

2. Translation and twisting functors

The action of the center $\mathcal{Z}$ of $\mathcal{U}(\mathfrak{g})$ gives a block decomposition

$$\mathcal{O} = \bigoplus_{\chi \in \mathrm{Max}\,\mathcal{Z}} \mathcal{O}_\chi, \tag{2.1}$$

where the blocks are indexed by maximal ideals of the center of $\mathcal{Z}$ and where $\mathcal{O}_\chi$ contains all modules annihilated by some power of $\chi \in \mathrm{Max}\,\mathcal{Z}$. The Harish-Chandra isomorphism ([Dix96, 7.4.6] shifted by ρ) defines a natural isomorphism ξ between the set of maximal ideals of $\mathcal{Z}$ and the orbits of $\mathfrak{h}^*$ under the dot-action of W, which are in bijection to the set of dominant weights. Under these bijections the decomposition (2.1) coincides with

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*_{\mathrm{dom}}} \mathcal{O}_\lambda,$$

where objects in $\mathcal{O}_\lambda$ have only composition factors of the form $L(w \cdot \lambda)$ with $w \in W$.

Let λ, μ be dominant such that $\lambda - \mu$ is integral. The *translation functor* $\theta_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is defined on objects $M \in \mathcal{O}_\lambda$ by $\theta_\lambda^\mu(M) = \mathrm{pr}_\mu(M \otimes E(\mu - \lambda))$, where $E(\mu - \lambda)$ denotes the finite dimensional $\mathfrak{g}$-module with extremal weight $\mu - \lambda$ and pr_μ is the projection onto the block $\mathcal{O}_\mu$. By definition, these functors are exact and θ_λ^μ is left and right adjoint to θ_μ^λ.

Let s be a simple reflection. Let λ be integral and regular and choose μ such that $\lambda - \mu$ is integral and $W_\mu = \{e, s\}$. We denote by θ_s the *translation through the s-wall*; i.e., $\theta_s = \theta_\mu^\lambda \theta_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$. Up to a natural isomorphism this functor is independent of the choice of μ. (For details see [Jan79], [Jan83], [BG80].)

Twisting functors T_w. For each $w \in W$ we define a *twisting functor* $T_w : \mathcal{O} \to \mathcal{O}$ as follows: Let $\mathfrak{n}_w = \mathfrak{n}_- \cap w^{-1}(\mathfrak{n}^+)$ and let $N_w = \mathcal{U}(\mathfrak{n}_w)$ be its universal enveloping algebra. We consider $\mathfrak{g}$ as a $\mathbb{Z}$-graded Lie algebra such that $\mathfrak{g}_1 = \bigoplus \mathfrak{g}_\alpha$, where the sum runs over all simple roots α. This induces uniquely a grading on N_w. Let $N_w^{\circledast} = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathbb{C}}\big((N_w)_i, \mathbb{C}\big)$ be its graded dual, i.e., $(N_w^{\circledast})_i = ((N_w)_{-i})^*$. With $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, the corresponding semi-infinite $\mathcal{U}$-bimodule S_w is then defined as

$$S_w = \mathcal{U} \otimes_{N_w} N_w^{\circledast}.$$

That this is a $\mathcal{U}$-*bi*module is not obvious (for a proof see [And03] and [Ark, 2.1.10], or the special case [Soe98, Theorem 1.3]); moreover, $S_w \cong N_w^{\circledast} \otimes_{N_w} \mathcal{U}$ as right $\mathcal{U}$-module. For a simple reflection s corresponding to α let $\mathcal{U}_{(s)}$ be the Ore localization of $\mathcal{U}$ at the set $\{1, x_{-\alpha}^i \mid i \in \mathbb{N}\}$. It contains naturally $\mathcal{U}$ as a subalgebra. There is a $\mathcal{U}$-bimodule isomorphism ([Ark], compare [AL02, Theorem 6.1])

$$\mathcal{U}_{(s)}/\mathcal{U} \tilde{\to} S_s. \tag{2.2}$$

The twisting functor T_w corresponding to $w \in W$ on the category of $\mathfrak{g}$-modules is given on objects by

$$M \quad \longmapsto \quad \phi_w(S_w \otimes_{\mathcal{U}} M),$$

and on morphisms by

$$f \longmapsto 1 \otimes f,$$

where ϕ_w means that the action of $\mathfrak{g}$ is twisted by an automorphism of $\mathfrak{g}$ corresponding to w. Up to isomorphism the functor is independent of this particular choice.

The main properties of twisting functors are given by the following

Lemma 2.1. *Let $w \in W$.*

(1) T_w is right exact.
(2) $T_{ws} \cong T_w T_s$ if $ws > w$ and s is a simple reflection.
(3) T_w preserves $\mathcal{O}$ and even each individual block $\mathcal{O}_\lambda$ for any $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$. In fact, $[T_w \Delta(\lambda)] = [\Delta(w \cdot \lambda)]$ in the Grothendieck group of $\mathcal{O}$.
(4) T_w commutes with tensoring with finite dimensional $\mathfrak{g}$-modules E, i.e., there is an isomorphism of functors $T_w(\bullet \otimes E) \cong (\bullet \otimes E)T_w$.
(5) T_w commutes with translation functors.

Proof. The first statement is clear by definition. For (2) and (3) see [AL02]. Statement (5) is then clear from (3) and (4). We prove (4) in a more general context in Theorem 3.2. (It is stated without a complete proof in [AL02]). □

Because of the second statement of the previous lemma we restrict ourselves to study the functor T_s for any simple reflection s. Moreover, we mainly consider its restriction to a regular integral block, say $\mathcal{O}_0$. Therefore, we also write $\Delta(x)$, $L(x)$, $\nabla(x)$, etc. instead of $\Delta(x \cdot 0)$, $L(x \cdot 0)$, $\nabla(x \cdot 0)$ respectively.

In [AL02] is described what happens to Verma modules when applying T_w. In the easiest case, where $x \in W$ and s is a simple reflection such that $sx > x$, it is (see [AL02, Lemma 6.2])

$$T_s \Delta(x) \cong \Delta(sx). \tag{2.3}$$

As seen in [AL02], twisted Verma modules (e.g., their socle and radical filtrations) are not well understood in general. However, it is known ([AL02, 6.3]) that for $sx > x$ they fit into a four step exact sequence of the form

$$0 \to \Delta(sx) \longrightarrow \Delta(x) \longrightarrow T_s\Delta(sx) \longrightarrow T_s\Delta(x) \to 0. \tag{2.4}$$

First properties of T_s and $\mathcal{L}T_s$. Recall that category $\mathcal{O}$ has enough projectives. Hence we may consider the left derived functors $\mathcal{L}_iT_w$ of T_w. It is clear from Lemma 2.1 that $\mathcal{L}_iT_w$ preserves $\mathcal{O}_\lambda$ for any $i \geq 0$ and $\lambda \in \mathfrak{h}^*_{\text{dom}}$. Likewise, Lemma 2.1 (4) and (5) generalize to all $\mathcal{L}_iT_w$. If we look at the functor $\mathcal{L}T_s$ for a simple reflection s, the situation becomes quite easy:

Theorem 2.2. *We have $\mathcal{L}_iT_s = 0$ for any $i > 1$ and any simple reflection s. If $\lambda \in \mathfrak{h}^*$, then $\mathcal{L}_iT_w\Delta(\lambda) = 0$ for $i > 0$ and $w \in W$.*

Proof. Since $S_s \cong N_s^{\circledast} \otimes_{N_s} \mathcal{U}$, the functor T_s is exact on N_s-free modules. Hence it is exact on Verma modules and on modules with a Verma flag. Since $N_s = \mathbb{C}[x_{-\alpha}]$ is a principal ideal domain for any simple reflection s, the functor T_s is even exact on submodules of projective modules. Let $M \in \mathcal{O}$ and let $P \twoheadrightarrow M$ be its projective cover with kernel K. By definition of $\mathcal{L}T_s$ we get $\mathcal{L}_iT_sM = \mathcal{L}_{i-1}T_sK$ for $i-1 > 0$. Since K is a submodule of a projective object in $\mathcal{O}$, the latter vanishes and the first part of the theorem follows.

Let $w \in W$. Since $S_w \cong N_w^{\circledast} \otimes_{N_w} \mathcal{U}$, the functor T_w is exact on N_w-free modules, in particular on modules with Verma flag. Therefore, $T_wP_\bullet \twoheadrightarrow T_w\Delta(\lambda)$ is exact for any projective resolution $P_\bullet \twoheadrightarrow \Delta(\lambda)$. The second statement of the theorem follows. □

For dual Verma modules the situation is very nice:

Theorem 2.3. *Let $x \in W$ and s be a simple reflection. There are isomorphisms of $\mathfrak{g}$-modules*

$$T_s\nabla(x) \cong \begin{cases} \nabla(x) & \text{if } x < sx, \\ \nabla(sx) & \text{if } x > sx; \end{cases}$$

and for $i > 0$,

$$\mathcal{L}_iT_s\nabla(x) \cong \begin{cases} K_{x,sx} & \text{if } x < sx \text{ and } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $K_{x,sx}$ denotes the kernel of the (unique up to a scalar) nontrivial (surjective) map $\nabla(x) \to \nabla(sx)$ in the case $x < sx$.

Proof. Let us start to prove both parts by (descending) induction on the length of x. If $x = w_o$, then $\nabla(w_o)$ is self-dual, hence $\mathcal{L}_1T_s\nabla(w_o) = 0$ by Theorem 2.2, and $T_s\nabla(w_o) = M^s(sw_o) \cong M^{w_o}(sw_o) \cong \nabla(sw_o)$ by [AL02, Lemma 6.2] (using the notations defined there).

Now let $x \neq w_o$ and choose a simple reflection $t \in W$ such that $xt > x$. Translation through the wall gives a short exact sequence of the form

$$0 \to \nabla(xt) \xrightarrow{f} \theta_t\nabla(xt) \longrightarrow \nabla(x) \to 0, \tag{2.5}$$

where f is the adjunction morphism and therefore we get, using Theorem 2.2 and Lemma 2.1 (5), a long exact sequence of the form

$$\begin{aligned} & 0 \to \mathcal{L}_1 T_s \nabla(xt) \longrightarrow \theta_t \mathcal{L}_1 T_s \nabla(xt) \longrightarrow \mathcal{L}_1 T_s \nabla(x) \\ & \longrightarrow T_s \nabla(xt) \xrightarrow{T_s f} \theta_t T_s \nabla(xt) \longrightarrow T_s \nabla(x) \to 0. \end{aligned} \tag{2.6}$$

Note, that $T_s f$ is always the adjunction morphism, since we have naturally (for any $M \in \mathcal{O}_0$, and $\theta_{on} = \theta_0^\lambda$, where $W_\lambda = \{e, t\}$)

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}}(M, \theta_t M) &= \mathrm{Hom}_{\mathcal{O}}(\theta_{on} M, \theta_{on} M) \overset{f \mapsto T_s f}{\longrightarrow} \mathrm{Hom}_{\mathcal{O}}(T_s \theta_{on} M, T_s \theta_{on} M) \\ &= \mathrm{Hom}_{\mathcal{O}}(\theta_{on} T_s M, \theta_{on} T_s M) = \mathrm{Hom}_{\mathcal{O}}(T_s M, \theta_t T_s M). \end{aligned}$$

Under this chain the adjunction map goes to the adjunction map.

Consider first the case $sxt < xt$. By induction hypotheses, (2.6) is of the form

$$0 \to \mathcal{L}_1 T_s \nabla(x) \longrightarrow \nabla(sxt) \xrightarrow{T_s f} \theta_t \nabla(sxt) \longrightarrow T_s \nabla(x) \to 0.$$

If $sxt > sx$ (i.e., $sx < x$), then $T_s f$ is injective with cokernel $\nabla(sx)$, hence $\mathcal{L}_1 T_s \nabla(x) = 0$ and $T_s \nabla(x) \cong \nabla(sx)$.

If $sxt < sx$, then $T_s f$ has to be the composite of the surjection $\nabla(sxt) \twoheadrightarrow \nabla(sx)$ between two 'neighboring' dual Verma modules and the adjunction morphism $\nabla(sx) \hookrightarrow \theta_t \nabla(sxt)$; hence $\mathcal{L}_1 T_s \nabla(x) \cong K_{sxt,sx}$ and $T_s \nabla(x) \cong \theta_t \nabla(sxt)/\nabla(sx) \cong \nabla(sxt)$. Our assumptions force $sxt = x$ by the Exchange Condition of W (see [Hum90]). Hence we have proved the theorem in the case $sxt < xt$.

Consider now the case $sxt > xt$. We have $l(sx) \geq l(sxt) - 1 = l(xt) = l(x) + 1$, i.e., $sx > x$. On the other hand, $T_s \nabla(xt) \cong \nabla(xt)$ by induction hypotheses, hence the adjunction morphism $T_s f$ is injective with cokernel $\nabla(x)$; i.e., $T_s \nabla(x) \cong \nabla(x)$ and the first terms of (2.6) provide by induction hypotheses a short exact sequence of the form

$$0 \to K_{xt,sxt} \xrightarrow{a} \theta_t K_{xt,sxt} \longrightarrow \mathcal{L}_1 T_s \nabla(x) \to 0, \tag{2.7}$$

where a is the restriction of the adjunction morphism. Consider the commutative diagram with exact rows and surjection p:

$$\begin{array}{ccccc} \nabla(xt) & \xrightarrow{adj} & \theta_t \nabla(xt) & \longrightarrow\!\!\!\!\!\rightarrow & \nabla(x) \\ \downarrow p & & \downarrow \theta_t p & & \downarrow \\ \nabla(sxt) & \xrightarrow{adj} & \theta_t \nabla(sxt) & \longrightarrow\!\!\!\!\!\rightarrow & \nabla(sx) \end{array}$$

By definition of $K_{xt,sxt}$ the kernel-cokernel sequence is of the form

$$0 \to K_{xt,sxt} \xrightarrow{a} \theta_t K_{xt,sxt} \longrightarrow K_{x,sx} \to 0.$$

By comparing it with (2.7), it follows that $\mathcal{L}_1 T_s \nabla(x) \cong K_{x,sx}$. Now, we proved all the statements concerning $\mathcal{L}_i T_s$ for $i \leq 1$. The vanishing of $\mathcal{L}_i T_s$ for $i > 1$ is Theorem 2.2. □

3. Tensoring with finite dimensional modules

To prove that the twisting functors behave well with respect to tensoring with finite dimensional modules we need a comultiplication on the 'completion' of the localization of $\mathcal{U}$.

Recall that $\mathcal{U}$ is a Hopf algebra with comultiplication $\Delta(u) = 1 \otimes u + u \otimes 1$, antipode $S(u) = -u$ and counit $c(u) = 0$ for elements $u \in \mathfrak{g}$. Note that $\Delta(u^n) = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} \otimes u^k$, for $u \in \mathcal{U}$.

We can consider $\mathcal{U}_{(s)}$ (and hence $\mathcal{U}_{(s)} \otimes \mathcal{U}_{(s)}$) as a left $\mathbb{C}[(x_{-\alpha})^{-1}]$-module. We denote by $\mathcal{U}_{(s)} \hat{\otimes} \mathcal{U}_{(s)}$ its extension to a module over the ring of formal power series $\mathbb{C}[[(x_{-\alpha})^{-1}]]$. There is an obvious extension of the algebra structure of $\mathcal{U}_{(s)} \otimes \mathcal{U}_{(s)}$ to $\mathcal{U}_{(s)} \hat{\otimes} \mathcal{U}_{(s)}$. The following result defines a comultiplication map from $\mathcal{U}_{(s)}$ into the completion $\mathcal{U}_{(s)} \hat{\otimes} \mathcal{U}_{(s)}$ of $\mathcal{U}_{(s)} \otimes \mathcal{U}_{(s)}$:

Lemma 3.1. *Let $s = s_\alpha$ be a simple reflection. The following map defines an algebra homomorphism*

$$\begin{array}{rcl} \tilde{\Delta} : \mathcal{U}_{(s)} & \longrightarrow & \mathcal{U}_{(s)} \hat{\otimes} \mathcal{U}_{(s)} \\ y^{-n} u & \mapsto & \Big(\sum_{k \geq 0} (-1)^k \binom{k+n-1}{k} y^{-n-k} \otimes y^k \Big) \Delta(u), \end{array}$$

for $y = x_{-\alpha}$ and any $u \in \mathcal{U}$.

Proof. Direct calculations show that the map is well-defined and defines an algebra homomorphism. □

Let $\mathcal{E}$ denote the category of finite dimensional $\mathfrak{g}$-modules.

Theorem 3.2. *Fix $w \in W$. There is a family $\{\mathrm{t}_E\}_{E \in \mathcal{E}}$ of isomorphisms of functors $\mathrm{t}_E : T_w \circ (\bullet \otimes E) \longrightarrow (\bullet \otimes E) \circ T_w$ such that the following diagrams commute:*

1) *For any E, $F \in \mathcal{E}$:*

$$\begin{array}{ccc} T_w(M \otimes E \otimes F) & \xrightarrow{\mathrm{t}_F(M \otimes E)} & T_w(M \otimes E) \otimes F \\ & \searrow^{\mathrm{t}_{E \otimes F}(M)} & \downarrow{\scriptstyle \mathrm{t}_E(M) \otimes \mathrm{id}} \\ & & T_w(M) \otimes E \otimes F \end{array}$$

2) *For any $E \in \mathcal{E}$:*

$$\begin{array}{ccc} T_w(M \otimes E \otimes E^*) & \xrightarrow{\mathrm{t}_{E \otimes E^*}(M)} & T_w(M) \otimes E \otimes E^* \\ \uparrow{\scriptstyle T_w(\mathrm{id} \otimes \overline{\mathrm{ev}})} & & \uparrow{\scriptstyle \mathrm{id} \otimes \overline{\mathrm{ev}}} \\ T_w(M \otimes \mathbb{C}) & \xrightarrow[\mathrm{t}_{\mathbb{C}}(M)]{} & T_w(M) \otimes \mathbb{C}, \end{array}$$

where $\overline{\mathrm{ev}} : \mathbb{C} \to E \otimes E^*$ is given by $1 \mapsto \sum_{i=1}^{d} e_i \otimes e_i^*$ for a fixed basis $(e_i)_{1 \leq i \leq d}$ of E with dual basis $(e_i^*)_{1 \leq i \leq d}$.

Proof. By Lemma 2.1 it is enough to consider the case where $w = s$ is a simple reflection. Let $M \in \mathcal{O}_0$ and let E be any finite dimensional $\mathfrak{g}$-module. Let s be the simple reflection corresponding to α. Set $y = x_{-\alpha}$. We consider the following map:

$$\begin{array}{rcl} \mathcal{U}_{(s)} \otimes_{\mathcal{U}} (M \otimes E) & \longrightarrow & (\mathcal{U}_{(s)} \otimes_{\mathcal{U}} M) \otimes E \\ y^{-n} \otimes (m \otimes e) & \longmapsto & \sum_{k \geq 0} (-1)^k \binom{n+k-1}{k} y^{-n-k} \otimes m \otimes y^k e. \end{array}$$

By Lemma 3.1 this map is well defined and a $\mathcal{U}$-morphism. It is even an isomorphism. To see this, one has to choose r, $a \in \mathbb{N}$ such that y^a annihilates E and

$(r-1)a \geq n-1$. Then $ar-n-k > ar-n-a \geq -1$ for any $k < a$. We can define a map in the opposite direction by $(y^{-n} \otimes m) \otimes e \mapsto y^{-ar} \otimes \sum_{k \geq 0} \binom{ar}{k} y^{ar-n-k} m \otimes y^k e$. Direct calculations show that this map does not depend on the actual choice of a and r and that it defines an inverse.

We check the commutativity of the first diagram: For $m \in M$, $e \in E$ and $f \in F$, the diagonal map is given by

$$\begin{aligned} y^{-n} \otimes m \otimes e \otimes f \quad &\mapsto \quad \sum_{k \geq 0} (-1)^k \binom{n+k-1}{k} y^{-n-k} \otimes m \otimes y^k (e \otimes f) \\ &= \quad \sum_{k \geq 0} \sum_{j \geq 0} (-1)^k \binom{n+k-1}{k} \binom{k}{j} y^{-n-k} \otimes m \otimes y^{k-j} e \otimes y^j f. \end{aligned} \tag{3.1}$$

On the other hand, the composite of the horizontal and vertical map is given by

$$\begin{aligned} y^{-n} \otimes m \otimes e \otimes f &\mapsto \sum_{j \geq 0} (-1)^j \binom{n+j-1}{j} y^{-n-j} \otimes (m \otimes e) \otimes y^j f \\ \mapsto \quad & \sum_{s \geq 0} \sum_{j \geq 0} (-1)^j \binom{n+j-1}{j} (-1)^s \binom{n+j+s-1}{s} y^{-n-j-s} \otimes m \otimes y^s e \otimes y^j f \\ = \quad & \sum_{k \geq 0} \sum_{j \geq 0} (-1)^k \binom{n+j-1}{j} \binom{n+k-1}{k-j} y^{-n-k} \otimes m \otimes y^{k-j} e \otimes y^j f, \end{aligned}$$

(by taking $k = j + s$). Since $\binom{n+j-1}{j}\binom{n+k-1}{k-j} = \binom{n+k-1}{k}\binom{k}{j}$ the diagonal map and the composition are the same and the diagram commutes.

For the second diagram we use formula (3.1) for $F = E^*$. We start with an element $y^{-n} \otimes (m \otimes 1) \in T_s(M \otimes \mathbb{C})$. By definition of the vertical map this element is mapped to $\sum_{i=1}^d y^{-n} \otimes m \otimes e_i \otimes e_i^*$. We apply the upper horizontal map to it. Since $\overline{\mathrm{ev}}$ is a morphism of $\mathfrak{g}$-modules, the resulting summands given by formula (3.1) are all trivial except for $k = 0$. Hence the composition maps $y^{-n} \otimes (m \otimes 1)$ to $\sum_{i=1}^d y^{-n} \otimes m \otimes e_i \otimes e_i^*$ which is obviously the same as what happens via $\left(\mathrm{id} \otimes \overline{\mathrm{ev}}\right) \circ t_{\mathbb{C}}(M)$. Therefore, the second diagram commutes. □

4. The adjoint functor G

In the following we want to describe the right adjoint functors of twisting functors. Note that since T_w is not left exact, there exists no left adjoint functor of T_w. The right $\mathcal{U}$-module structure on S_w defines a left $\mathcal{U}$-module structure on $\mathrm{Hom}_{\mathcal{U}}(S_w, \phi_w^{-1}(M))$ for any $\mathfrak{g}$-module M. Let $G_w(M)$ denote the maximal $\mathfrak{h}$-diagonalizable submodule of $\mathrm{Hom}_{\mathcal{U}}(S_w, \phi_w^{-1}(M))$. Then $M \mapsto G_w(M)$ defines a left exact endofunctor of $\mathcal{O}$ which is right adjoint to T_w. It is straightforward to check that for $f \in \mathrm{Hom}_{\mathcal{O}}(T_w M, N)$ we can define $\hat{f} \in \mathrm{Hom}_{\mathcal{O}}(M, G_w N)$ by $\hat{f}(m)(s) = f(s \otimes m)$, where $m \in M$, $s \in S_w$. Then $f \mapsto \hat{f}$ defines the desired adjunction. (The inverse map is given by $g \mapsto \check{g}$, where $\check{g}(s \otimes m) = f(m)(s)$.) Lemma 2.1 immediately implies that $G_{ws} \cong G_s G_w$ for any $w \in W$, $s \in S$ such that $ws > w$.

A surprisingly strong link between T_w and G_w is provided by the following

Theorem 4.1. *For any $w \in W$, there is an isomorphism of functors*

$$G_w \cong \mathrm{d}\, T_{w^{-1}}\, \mathrm{d}\,.$$

Proof. Let us assume for a moment that the theorem is true for all simple reflections. Let $w = s_1 \cdot \ldots \cdot s_r$ be a reduced expression. By Lemma 2.1, we have $T_w \cong T_{s_1} \cdot \ldots \cdot T_{s_r}$, hence its right adjoint is isomorphic to

$$\begin{aligned} G_{s_r}G_{s_{r-1}} \cdot \ldots \cdot G_{s_1} &\cong (\mathrm{d}\, T_{s_r}\, \mathrm{d})(\mathrm{d}\, T_{s_{r-1}}\, \mathrm{d}) \cdot \ldots \cdot (\mathrm{d}\, T_{s_1}\, \mathrm{d}) \\ &\cong \mathrm{d}\, T_{w^{-1}}\, \mathrm{d}\,. \end{aligned}$$

Now let s be a simple reflection. By extending ϕ_s to $\mathcal{U}$ we get an anti-automorphism $\tau = \phi_s \circ \sigma$ of $\mathcal{U}$ with inverse $\tau^{-1} = \sigma \circ \phi_s^{-1}$. We consider the following map:

$$\begin{aligned} \Psi_s : \mathrm{Hom}_{\mathcal{U}}\left(\mathcal{U}, \phi_s^{-1}(M)\right) &\longrightarrow \mathrm{d}\left(\phi_s(\mathcal{U} \otimes_{\mathcal{U}} \mathrm{d}(M))\right) \\ f &\longmapsto \hat{f}; \qquad \hat{f}(u \otimes m^*) = m^*(f(\tau(u))), \end{aligned}$$

for $u \in \mathcal{U}$ and $m^* \in M^*$. For $x \in \mathfrak{g}$, $u \in \mathcal{U}$ and $m^* \in M^*$ we get $\hat{f}(u \otimes xm^*) = xm^*(f(\tau(u))) = m^*(\sigma(x)f(\tau(u))) = m^*(f(\tau(x)\tau(u))) = m^*(f(\tau(ux))) = \hat{f}(ux \otimes m^*)$. Hence, $\hat{f}$ is well defined. Direct calculations also show that Ψ_s is a $\mathcal{U}$-morphism. The inverse of Ψ_s is given by $g \mapsto \check{g}$, where $m^*(\check{g}(u)) = g(\tau^{-1}(u) \otimes m^*)$. Altogether, Ψ_s is an isomorphism of vector spaces.

Since τ stabilizes the multiplicative set defining $\mathcal{U}_{(s)}$ (see (2.2)), Ψ_s defines a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}}\left(\mathcal{U}_{(s)}, \phi_s^{-1}(M)\right) \longrightarrow \mathrm{d}\left(\phi_s(\mathcal{U}_{(s)} \otimes_{\mathcal{U}} \mathrm{d}(M))\right).$$

Hence the theorem follows from the isomorphism (2.2). □

For $w \in W$ we denote by $\mathcal{R}G_w$ the right derived functor of G_w. We get a remark in [Ark] as

Corollary 4.2. *Let $w \in W$. The left derived functor $\mathcal{L}T_w$ of T_w defines an auto-equivalence of the bounded derived category $\mathcal{D}^b(\mathcal{O}_0)$.*

Proof. First let $w = s$ be a simple reflection. By Theorem 2.2 and (2.3) we have $\mathcal{L}T_s\Delta(e) \cong \Delta(s)$ and $\mathcal{R}G_s(\Delta(s)) \cong \mathrm{d}\,\mathcal{L}T_s\nabla(s) \cong \Delta(e)$ by Theorem 4.1 and Theorem 2.3. Hence $\mathcal{R}G_s\mathcal{L}T_sP \cong P$ for any projective module $P \in \mathcal{O}_0 \subset \mathcal{D}^b(\mathcal{O}_0)$. Any finite complex of modules in $\mathcal{O}_0$ is quasi-isomorphic to a finite complex of projective modules, hence $\mathcal{R}G_s\mathcal{L}T_s \cong \mathrm{id}$.

The previous theorem, Theorem 2.2 and formula (2.3) give $\mathcal{R}G_s\nabla(e) \cong \nabla(s)$ and $\mathcal{L}T_s(\nabla(s)) \cong \nabla(e)$ by the previous theorem and Theorem 2.3. Hence $\mathcal{L}T_s\mathcal{R}G_sI \cong I$ for any injective module $I \in \mathcal{O}_0 \subset \mathcal{D}^b(\mathcal{O}_0)$. Any finite complex of modules in $\mathcal{O}_0$ is quasi-isomorphic to a finite complex of injective modules, hence $\mathcal{L}T_s\mathcal{R}G_s \cong \mathrm{id}$. Therefore, $\mathcal{L}T_s$ defines an auto-equivalence of $\mathcal{D}^b(\mathcal{O}_0)$. By Lemma 2.1 we have $\mathcal{L}T_{ws} \cong \mathcal{L}T_w\mathcal{L}T_s$ for any $w \in W$ and $s \in S$, such that $ws > w$. Therefore, the statement follows for any $w \in W$. □

5. Some natural transformations

For any $x \in W$ and $s \in S$ we call $L(x)$ *s-finite* if $x < sx$, otherwise *s-free*. A module $M \in \mathcal{O}_0$ is called s-finite (s-free respectively) if all its composition factors are s-finite (s-free respectively). The following lemma characterizes s-finite simple modules using T_s:

Proposition 5.1. *Let $x \in W$. Then $T_sL(x) \neq 0$ if and only if $x > sx$.*

Proof. If $x < sx$, we consider the inclusion $i : \Delta(sx) \hookrightarrow \Delta(x)$. Let Q be its cokernel, hence there is an exact sequence

$$T_s\Delta(sx) \xrightarrow{T_s i} T_s\Delta(x) \longrightarrow T_sQ \to 0.$$

The sequence (2.4) shows the existence of a surjection in $\mathrm{Hom}_{\mathcal{O}}(T_s\Delta(sx), T_s\Delta(x))$. Now the latter is (via T_s) isomorphic to $\mathrm{Hom}_{\mathcal{O}}(\Delta(sx), \Delta(x)) = \mathbb{C}$. That is $T_s i \neq 0$ and therefore surjective, i.e., $T_sQ = 0$. Hence $T_sL(x) = 0$, as $L(x)$ is a quotient of Q.

If $x > sx$, then

$$0 \neq \mathrm{Hom}_{\mathcal{O}}(\Delta(x), L(x)) = \mathrm{Hom}_{\mathcal{O}}(T_s\Delta(sx), L(x)) = \mathrm{Hom}_{\mathcal{O}}(\Delta(sx), G_sL(x)),$$

hence $G_sL(x) \cong \mathrm{d}\, T_sL(x) \neq 0$. □

Corollary 5.2. *The head of T_sM is s-free for any $M \in \mathcal{O}_0$ and $s \in S$.*

Proof. The right adjoint functor G_s of T_s annihilates all s-finite simple modules by the previous proposition and Theorem 4.1. □

As for Verma modules we can describe at least some of the twisted projective modules:

Proposition 5.3. *Let $x \in W$, $s \in S$ such that $sx > x$; then $T_sP(sx) \cong P(sx)$. Dually, $G_sI(sx) \cong I(sx)$.*

Proof. By formula (2.3) $T_s\Delta(e) \cong \Delta(s)$ and therefore $T_sP(s) \cong T_s\theta_s\Delta(e) \cong \theta_sT_s\Delta(e) \cong P(s)$ since T_s commutes with θ_s. Now let $x = s_1 \cdot \ldots \cdot s_r$ be a reduced expression and $\theta = \theta_{s_1} \cdot \ldots \cdot \theta_{s_r}$. We get

$$T_s\theta P(s) \cong \theta T_sP(s) \cong \theta P(s) \twoheadrightarrow P(sx),$$

where the last map is a surjection. Hence $P(sx)$ is a direct summand of $T_s\theta P(s)$. Let $\theta P(s) \cong \bigoplus_{y>sy} P(y)^{n_y}$ for some $n_y \in \mathbb{N}$. (The condition on y comes from the fact that $\theta P(s) \cong T_s\theta P(s)$ has an s-free head by Corollary 5.2.) Hence $P(sx)$ is a direct summand of $T_sP(y)$ for some $y \in W$ such that $sy < y$, i.e., $P(sx) \cong T_sP(y)$, because of the indecomposability of $T_sP(y)$ (by Corollary 4.2 we have $\mathrm{End}_{\mathcal{O}}(T_sP) \cong \mathrm{End}_{\mathcal{O}}(P)$ for all projective modules $P \in \mathcal{O}$). This implies, with the formulas (2.4) and (2.3), the existence of a chain of surjections $P(sx) \twoheadrightarrow T_s\Delta(y) \twoheadrightarrow T_s\Delta(sy) \cong \Delta(y)$. Hence, $sx = y$ and the proposition follows. □

Proposition 5.4. *Let s be a simple reflection. Assume, there is a nontrivial homomorphism $\mathrm{can}_s : T_s \longrightarrow \mathrm{id}$ of endofunctors on $\mathcal{O}_0$ such that $\mathrm{can}_s(\Delta(e)) \neq 0$. Then, for any $M \in \mathcal{O}_0$, the cokernel of $\mathrm{can}_s(M)$ is the largest s-finite quotient of M.*

Proof. For $x \in W$ fix an inclusion $i_x : \Delta(x) \hookrightarrow \Delta(e)$.

We assume $\mathrm{can}_s(\Delta(e)) \neq 0$. By formula (2.3) we get that the image must be the submodule $\Delta(s)$. If $x \in W$ such that $sx > x$, then $\mathrm{can}_s(\Delta(e)) \circ T_si_x$ has image isomorphic to $\Delta(sx)$, hence the image of $\mathrm{can}_s(\Delta(x))$ must be the submodule isomorphic to $\Delta(sx)$ inside $\Delta(x)$, i.e., the map is injective. If $x > sx$, then for $j \in \mathrm{Hom}_{\mathcal{O}}(\Delta(x), \Delta(sx))$ the map T_sj is surjective (see proof of Lemma 5.1). Hence $\mathrm{im}(j \circ \mathrm{can}_s(\Delta(x)) = \mathrm{im}(\mathrm{can}_s(\Delta(sx)) \circ T_sj) \cong T_s\Delta(sx) \cong \Delta(x)$. This forces $\mathrm{can}_s(\Delta(sx))$ to be surjective. That is, $p \circ \mathrm{can}_s(\Delta(x)) = \mathrm{can}_s(L(x)) \circ T_sp$ is surjective, where p denotes a surjection from $\Delta(x)$ onto its head. Hence, $\mathrm{can}_s(L(x))$ is surjective.

Let M be s-finite. By induction on the length of M, Lemma 5.1 gives $T_sM = 0$, hence the statement of the proposition is true for M. Now let $M \in \mathcal{O}_0$ be arbitrary with maximal s-finite quotient Q. Let K be defined by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\quad i \quad} & M & \xrightarrow{\quad q \quad} & Q & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{can}_s(K)} & & \uparrow{\scriptstyle \mathrm{can}_s(M)} & & \uparrow{\scriptstyle \mathrm{can}_s(Q)} & & \\
 & & T_sK & \xrightarrow{\quad T_si \quad} & T_sM & \xrightarrow{\quad T_sq \quad} & T_sQ & \longrightarrow & 0.
\end{array}$$

Since $T_sQ = 0$ by assumption, we get $\operatorname{im} \mathrm{can}_s(M) \subseteq i(K)$. For the reverse inclusion it is enough to show that $\mathrm{can}_s(K)$ is surjective. By the maximality condition on Q, the head H of K is s-free. Let p be a surjection from K onto H. Since $\mathrm{can}_s(H) \circ T_sp = p \circ \mathrm{can}_s(K)$ and (as we already proved) $\mathrm{can}_s(H)$ is surjective, $\mathrm{can}_s(K)$ has to be surjective. Hence $\operatorname{im} \mathrm{can}_s(M) = i(K)$ and the proposition is proved. □

Remarks 5.5. Dualizing the properties of T_s described in Theorem 3.2 gives properties of completion functors as explained in [Jos83]. It is not clear whether these conditions in fact characterize completion functors. Assuming this, we would get that G_s is a completion functor, hence there is by definition a natural transformation from the identity functor to G_s and therefore also a homomorphism of functors as described in the previous proposition. The morphisms T_α^M and δT_α^M from [Jos83, 2.6] would then be exactly the adjunction morphisms of Proposition 5.6. On the other hand, O. Khomenko and V. Mazorchuk announced [KM] a different proof of the existence of a natural transformation as assumed in Proposition 5.4.

Adjunction and s-finiteness. The following two results would be an easy corollary of Proposition 5.4. We prove them without assuming the existence of a canonical morphism of functors (as assumed in Proposition 5.4):

Proposition 5.6. *Let s be a simple reflection. The morphism* $\mathrm{adj} : T_sG_s \mapsto \mathrm{id}$ *given by adjunction is injective on any object $M \in \mathcal{O}_0$. Moreover, the cokernel of* $\mathrm{adj}(M)$ *is the largest s-finite quotient of M. Dual statements hold for the adjunction morphism* $\mathrm{id} \to G_sT_s$.

Proof. Since $T_sG_s\nabla(e) \cong \nabla(e)$ and both functors commute with translations, $T_sG_s(I) \cong I$ via the adjunction morphism for all injective objects $I \in \mathcal{O}_0$. For $M \in \mathcal{O}_0$ let $M \overset{i}{\hookrightarrow} I \to J$ be an injective copresentation. We get an exact sequence

$$0 \to G_sM \xrightarrow{G_si} G_sI \longrightarrow Q \to 0, \tag{5.1}$$

where $Q \subseteq G_s(J)$. Now, $\mathcal{L}_1T_s(G_s\nabla(e)) = \mathcal{L}_1T_s\nabla(s) = 0$ by Theorem 2.3, hence $\mathcal{L}_1T_s(G_sJ) = 0$ which forces $\mathcal{L}_1T_sQ = 0$ by Theorem 2.2. Applying T_s to (5.1) gives therefore an exact sequence $0 = \mathcal{L}_1T_sQ \to T_sG_sM \xrightarrow{T_sG_si} T_sG_s(I)$, hence T_sG_si is injective. By definition of the adjunction morphism $i \circ \mathrm{adj}(M) = \mathrm{adj}(I) \circ T_sG_si$; hence a composite of injective maps, which implies the injectivity of $\mathrm{adj}(M)$.

Now let $I = I(sx)$ be an indecomposable injective object of $\mathcal{O}_0$ with s-free socle. By Proposition 5.3, $G_sI \cong I$. We calculate:

$$\begin{aligned}\dim \operatorname{Hom}_{\mathcal{O}}(T_sG_sM, I) &= \dim \operatorname{Hom}_{\mathcal{O}}(G_sM, G_sI) = \dim \operatorname{Hom}_{\mathcal{O}}(G_sM, I)\\ &= \dim \operatorname{Hom}_{\mathcal{O}}(T_s \operatorname{d} M, I) = \dim \operatorname{Hom}_{\mathcal{O}}(\operatorname{d} M, G_sI)\\ &= \dim \operatorname{Hom}_{\mathcal{O}}(\operatorname{d} M, I) = \dim \operatorname{Hom}_{\mathcal{O}}(M, I).\end{aligned}$$

Hence the cokernel of the adjunction morphism for M is s-finite. Since the head of T_sG_sM is s-free (Corollary 5.2), the cokernel is in fact the largest s-finite quotient of M. □

Remarks 5.7. Combining this proposition with Proposition 5.4 we see that, if the morphism $\mathrm{can}_s : T_s \to \mathrm{id}$ exists, then it factors through $\mathrm{adj} : T_sG_s \to \mathrm{id}$ and for each $M \in \mathcal{O}_0$ the resulting homomorphism $T_sM \to T_sG_sM$ is surjective.

Corollary 5.8. *Let $M \in \mathcal{O}_0$ and s be a simple reflection. Then $T_sM = 0$ if and only if M is s-finite.*

Proof. If M is s-finite, then $T_sM = 0$ by Lemma 5.1 and induction on the length of M. Assume $0 = T_sM = \operatorname{d} G_s \operatorname{d} M$, hence $G_s \operatorname{d} M = 0$ and therefore $T_sG_s \operatorname{d} M = 0$. By the previous proposition, $\operatorname{d} M$ is s-finite and therefore so is M. □

Corollary 5.9. *Let $M \in \mathcal{O}_0$ and s be a simple reflection.*

(1) $\mathcal{L}_1T_sM$ and $\mathcal{R}^1G_sM$ are s-finite.
(2) $\mathcal{L}_1T_s(G_sM) = 0 = \mathcal{R}^1G_s(T_sM)$.
(3) We have short exact sequences

$$0 \to T_sG_sM \longrightarrow M \longrightarrow \mathcal{L}_1T_s(\mathcal{R}^1G_sM) \to 0,$$

and

$$0 \to \mathcal{R}^1G_s(\mathcal{L}_1T_sM) \longrightarrow M \longrightarrow G_sT_sM \to 0.$$

Proof. We shall prove the first half of each statement. The second halves follow then by duality using Theorem 4.1.

(1) Note first that $\mathcal{L}_1T_s\nabla(e) = K_{e,s}$ by Theorem 2.3; and that $T_sK_{e,s} = 0$, because $\mathcal{L}_1T_s\nabla(s) = 0$ and $T_s\nabla(e) \cong \nabla(e) \cong T_s\nabla(s)$. Hence $K_{e,s}$ is s-finite by Corollary 5.8 and therefore so is $K_{e,s} \otimes E$ for any finite dimensional $\mathfrak{g}$-module E. On the other hand, M is isomorphic to a submodule of $\nabla(e)\otimes E$ for such an E and hence $\mathcal{L}_1T_sM$ is a submodule of $\mathcal{L}_1T_s(\nabla(e) \otimes E) \cong K_{e,s} \otimes E$. As a submodule of an s-finite module, $\mathcal{L}_1T_sM$ is s-finite.

(2) Take E as above with an injection $M \hookrightarrow \nabla(e) \otimes E$. The left exactness of G_s gives an injection $G_s(M) \hookrightarrow G_s(\nabla(e)\otimes E) \cong (G_s\nabla(e))\otimes E \cong \nabla(s)\otimes E$ by Theorem 3.2, formula (2.3) and Theorem 4.1. Hence $\mathcal{L}_1T_s(G_sM)$ is isomorphic to a submodule of $\mathcal{L}_1T_s(\nabla(s) \otimes E) \cong \mathcal{L}_1T_s(\nabla(s)) \otimes E = 0$ (by Theorem 2.3).

(3) Let Q be the maximal s-finite quotient of M. Then Proposition 5.6 gives us a short exact sequence of the form

$$0 \to T_sG_sM \longrightarrow M \longrightarrow Q \to 0.$$

Via previous statements (1) and (2) this shows that $\mathcal{R}^1G_sM \cong \mathcal{R}^1G_sQ$. That means, we are done if we prove the following
Claim: For any s-finite module $Q \in \mathcal{O}_0$ we have $Q \cong \mathcal{L}_1T_s(\mathcal{R}^1G_sQ)$.

To see this, we choose an embedding $Q \hookrightarrow \nabla(e) \otimes F$ for some finite dimensional $\mathfrak{g}$-module F. Let Q' be its cokernel, hence we have a short exact sequence

$$0 \to Q \longrightarrow \nabla(e) \otimes F \longrightarrow Q' \to 0.$$

Applying G_s and using Corollary 5.8, Theorem 4.1, and Theorem 2.3 give an exact sequence

$$0 \to G_s(\nabla(e) \otimes F) \longrightarrow G_s Q' \longrightarrow \mathcal{R}^1 G_s Q \to 0.$$

By applying T_s and adding the adjunction morphism adj from Proposition 5.6, we get the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_1 T_s(\mathcal{R}^1 G_s(Q)) & \longrightarrow & T_s G_s(\nabla(e) \otimes F) & \longrightarrow & T_s G_s Q' & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow \mathrm{adj} & & \downarrow \mathrm{adj} & & \\ 0 & \longrightarrow & Q & \longrightarrow & \nabla(e) \otimes F & \longrightarrow & Q' & \longrightarrow & 0 \end{array}$$

The first row is exact (by (2), (1) and Corollary 5.8). This induces an injective homomorphism g as indicated such that the diagram commutes. The middle map is an isomorphism by Proposition 5.6, hence g is also an isomorphism. The claim follows and therefore we are done.

□

6. Twisting simple modules

The following theorem describes $\mathcal{L}_1 T_s$ on simple modules. A description of $T_s L$, when L is s-free is given by Theorem 6.3.

Theorem 6.1.

$$\mathcal{L}_1 T_s L(x) \cong \begin{cases} L(x) & \text{if } x < sx, \\ 0 & \text{if } x > sx. \end{cases}$$

Proof. By Theorem 2.2, $\mathcal{L}_i T_s L(x) = 0$ for $i > 1$, hence $\mathcal{L}_1 T_s L(x)$ is a submodule of $\mathcal{L}_1 T_s \nabla(x)$ (using that $L(x)$ is a submodule of $\nabla(x)$). The latter vanishes if $x > xs$ by Theorem 2.3. Now let $x < sx$. Corollary 4.2 and Theorem 5.1 imply that $\mathrm{End}_{\mathcal{O}}(\mathcal{L}_1 T_s L(x)) \cong \mathrm{End}_{\mathcal{O}}(L(x))$, hence $\mathcal{L}_1 T_s L(x) \neq 0$. By Corollary 5.9 $\mathcal{L}_1 T_s L(x)$ is s-finite. Being a submodule of $\mathcal{L}_1 T_s \nabla(x) = K_{x,sx}$ (Theorem 2.3) it has simple socle $L(x)$. Let $L(z)$ be s-finite and assume $\mathrm{Hom}_{\mathcal{O}}(\mathcal{L}_1 T_s L(y), L(z)) \neq 0$. Then $0 \neq \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}_1 T_s L(y), \mathcal{L}_1 T_s L(z)) = \mathrm{Hom}_{\mathcal{O}}(L(y), L(z))$, by Corollary 4.2 and Lemma 5.1. Hence we proved, that $\mathcal{L}_1 T_s L(x)$ has simple socle and simple head $L(x)$. On the other hand, $L(x)$ occurs at most once as a composition factor of $T_s L(x)$ and $T_s L(x) \neq 0$. This proves the remaining part. □

The following consequence of Theorem 6.1 bares some resemblance to the Borel-Weil-Bott Theorem for cohomology of line bundles on flag varieties (cf. [Jan87, Corollary 5.5])

Corollary 6.2. *Let $w \in W$, then*

$$\mathcal{L}_i T_w L(e) \cong \begin{cases} L(e) & \text{if } i = l(w), \\ 0 & \text{if } i \neq l(w). \end{cases}$$

Proof. Note that $L(e)$ is s-finite for all $s \in S$. Hence the corollary is an immediate consequence of Theorem 6.1 and the fact that $T_{sw} \cong T_w T_s$ if $sw > w$. (Observe that since $T_s\Delta(e) \cong \Delta(s \cdot 0)$, we have that T_s transforms projective modules into T_w acyclic objects; i.e., we have the Grothendieck spectral sequence (see [McC85, Theorem 10.7]) for the composition $T_w T_s$.) □

Theorem 6.3. *Let $L \in \mathcal{O}_0$ be a simple s-free module. Then:*

(1) T_sL has simple head L.
(2) T_sL is an indecomposable extension of L by some s-finite submodule U.
(3) $\dim \mathrm{Hom}_{\mathcal{O}}(L', T_sL) = \dim \mathrm{Ext}^1_{\mathcal{O}}(L', L)$ for any simple object $L' \in \mathcal{O}$.
(4) $\mathrm{soc}(U) \cong \mathrm{hd}(U)$.

Proof. Let P be an indecomposable projective module with s-free head. By Corollary 5.3, we have $T_sP \cong P$, hence $\dim \mathrm{Hom}_{\mathcal{O}}(P, T_sL) = \dim \mathrm{Hom}_{\mathcal{O}}(T_sP, T_sL) = \dim \mathrm{Hom}_{\mathcal{O}}(P, L)$ by Corollary 4.2, Theorem 2.2 and Theorem 6.1. Hence, L is the only s-free composition factor occurring in T_sL and its multiplicity is one. Since T_s, hence G_s, annihilates all s-finite objects, the head of T_sL has to be s-free, so it is isomorphic to L. Altogether, T_sL is an indecomposable extension of L by some s-finite U.

Let L' be an s-finite simple module in $\mathcal{O}_0$. Using again Corollary 4.2 and Theorem 6.1 we get

$$\begin{aligned} \dim \mathrm{Hom}_{\mathcal{O}}(L', T_sL) &= \dim \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}_1 T_s L', T_sL) \\ &= \dim \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_0)}(\mathcal{L}T_sL', \mathcal{L}T_sL[1]) \\ &= \dim \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_0)}(L', L[1]) \\ &= \dim \mathrm{Ext}^1_{\mathcal{O}}(L', L). \end{aligned}$$

(Here, $[\cdot]$ denotes the 'translation functor' on $\mathcal{D}^b(\mathcal{O})$).

To prove (4) we consider the exact sequence

$$0 \to U \longrightarrow T_sL \longrightarrow L \to 0. \tag{6.1}$$

For any s-finite simple object L' this gives an exact sequence

$$0 = \mathrm{Hom}_{\mathcal{O}}(T_sL, L') \longrightarrow \mathrm{Hom}_{\mathcal{O}}(U, L') \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(L, L') \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(T_sL, L') \to \cdots$$

On the other hand, $\mathrm{Ext}^i_{\mathcal{O}}(T_sL, L') \cong \mathrm{Ext}^{i-1}_{\mathcal{O}}(L, \mathcal{R}^1G_sL') \cong \mathrm{Ext}^{i-1}_{\mathcal{O}}(L, L')$ for any $i > 0$ by Corollary 4.2, Theorem 4.1 and Theorem 6.1. In particular, the last term in the sequence above vanishes and implies $\mathrm{Hom}_{\mathcal{O}}(U, L') \cong \mathrm{Ext}^1_{\mathcal{O}}(L, L')$. Comparing this with (3) proves statement (4). □

7. The Kazhdan-Lusztig conjecture

In this section we demonstrate how our results on twisting functors reduce the Kazhdan-Lusztig conjecture [KL79] on the characters of irreducible modules in $\mathcal{O}$ to a rather innocent looking statement; see Claim 7.1 below. The arguments for this reduction are borrowed from [Vog79], [And86] and [CPS93] with the necessary modifications and simplifications called for by our approach. Of course, it is well known that the Kazhdan-Lusztig conjecture holds; see [BB93] and [BK81].

An implication of the Kazhdan-Lusztig conjecture. By Theorem 6.3 it is clear that the existence of a nontrivial extension between simple modules with adjacent highest weights (a well-known consequence of the Kazhdan-Lusztig conjecture) gives the following

Claim 7.1. For all $x \in W$ with $x < w_0$ there exists $s \in S$ such that $L(x)$ is a submodule of $T_sL(sx)$.

We denote the kernel of a surjective homomorphism $T_sL(sx) \twoheadrightarrow L(sx)$ by $U_s(sx)$ (see Theorem 6.3). It is clear from weight considerations that $x \cdot 0$ is the unique highest weight in $U_s(sx)$ and that it has multiplicity 1. Since the socle of $U_s(sx)$ is isomorphic to its head by Theorem 6.3, we see that the above statement is equivalent to the following

Claim 7.2. For all $x \in W$ with $x < w_0$ there exists $s \in S$ such that $L(x)$ is a summand of $U_s(sx)$.

The following lemma describes assumptions under which the equivalent Claims 7.1 and 7.2 hold

Lemma 7.3. *Claim 7.1 (or Claim 7.2) is true if for any $x \in W$ there exists $s \in S$ such that $sx > x$ and one of the following (equivalent) statements hold:*

(1) $\operatorname{hd}\operatorname{rad}\Delta(x)$ *is not s-finite.*
(2) $\operatorname{Hom}_{\mathcal{O}}(T_s \operatorname{rad}\Delta(x), T_sL(sx)) \neq 0$.

Proof. Let $x \in W$ and $s \in S$ such that $sx > x$. Since $\operatorname{Hom}_{\mathcal{O}}(T_s \operatorname{rad}\Delta(x), T_sL(sx)) \cong \operatorname{Hom}_{\mathcal{O}}(\operatorname{rad}\Delta(x), L(sx))$ the second property implies the first. Consider the short exact sequence $\Delta(sx) \hookrightarrow \Delta(x) \twoheadrightarrow Q$ with s-finite quotient Q. In particular, we get a short exact sequence of the form $\Delta(sx) \hookrightarrow \operatorname{rad}\Delta(x) \twoheadrightarrow Q'$, with s-finite quotient Q'. Hence, $L(sx)$ has to occur (as the only s-free candidate) as the composition factor in the head of $\operatorname{rad}\Delta(x)$. Since $\operatorname{Hom}_{\mathcal{O}}(\Delta(x), L(sx)) = 0 = \operatorname{Ext}^1_{\mathcal{O}}(\Delta(x), L(sx))$ by weight considerations, we get $\operatorname{Hom}_{\mathcal{O}}(\operatorname{rad}\Delta(x), L(sx)) \cong \operatorname{Ext}^1_{\mathcal{O}}(L(x), L(sx))$. By assumption this does not vanish and gives with Theorem 6.3 Claim 7.1. □

Consequences under the assumption of Claim 7.1. The rest of this section is devoted to proving that Claim 7.1 (or Claim 7.2) implies the Kazhdan-Lusztig conjecture.
The first consequence of the statement in Claim 7.1 is the following even-odd vanishing result.

Proposition 7.4. *Assume Claim 7.1. If $\operatorname{Ext}^i_{\mathcal{O}}(\Delta(y), L(x)) \neq 0$ for some $i \in \mathbb{N}$ and $x, y \in W$, then $i \equiv l(y) - l(x) (\operatorname{mod} 2)$.*

Proof. We shall proceed by descending induction on $x \in W$. If $x = w_0$, then $L(x) = \nabla(w_0)$ and we have $\operatorname{Ext}^i_{\mathcal{O}}(\Delta(y), \nabla(w_0)) = 0$ unless $i = 0$ and $y = w_0$.

So assume now that $x < w_0$ and pick s as in Claim 7.1. Then $sx > x$ because otherwise $T_sL(sx) = 0$ by Theorem 6.1. Hence by induction the proposition holds for sx.

First case: $sy < y$: The short exact sequence

$$0 \to U_s(sx) \to T_sL(sx) \to L(sx) \to 0$$

gives rise to the long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}_{\mathcal{O}}\big(\Delta(y), L(sx)\big) \to \operatorname{Ext}^{i}_{\mathcal{O}}\big(\Delta(y), U_s(sx)\big) \to \operatorname{Ext}^{i}_{\mathcal{O}}\big(\Delta(y), T_sL(sx)\big) \to \cdots .$$

The assumption $sy < y$ gives $\Delta(y) \cong T_s\Delta(sy)$ by formula (2.3) and hence we get $\mathrm{Ext}^i_{\mathcal{O}}(\Delta(y), T_sL(sx)) \cong \mathrm{Ext}^i_{\mathcal{O}}(T_s\Delta(sy), T_sL(sx)) \cong \mathrm{Ext}^i_{\mathcal{O}}(\Delta(sy), L(sx))$. Our induction hypothesis therefore shows that $\mathrm{Ext}^i_{\mathcal{O}}(\Delta(y), U_s(sx)) = 0$ if $i \not\equiv l(y) - l(x)(\mathrm{mod}\, 2)$. By Claim 7.2 this means that also $\mathrm{Ext}^i_{\mathcal{O}}(\Delta(y), L(x)) = 0$ for such i.

Second case: $sy > y$: Note that since $sx > x$, we have $G_sL(x) = 0$ and $\mathcal{R}^1G_sL(x) \cong L(x)$ by Theorem 4.1, Lemma 5.1 and Theorem 6.1. Therefore, $\mathrm{Ext}^i_{\mathcal{O}}(\Delta(y), L(x)) \cong \mathrm{Ext}^i_{\mathcal{O}}(\Delta(y), \mathcal{R}^1G_sL(x)) \cong \mathrm{Ext}^{i+1}_{\mathcal{O}}(T_s\Delta(y), L(x))$ (the last equality by Theorem 6.1). Now the assumption $sy > y$ implies $T_s\Delta(y) \cong \Delta(sy)$ (formula (2.3)) and hence $\mathrm{Ext}^i_{\mathcal{O}}(\Delta(y), L(x)) \cong \mathrm{Ext}^{i+1}_{\mathcal{O}}(\Delta(sy), L(x)) = 0$ for $i \not\equiv l(y) - l(x)(\mathrm{mod}\, 2)$ by the first case treated above. □

Corollary 7.5. *Assume Claim 7.1. If* $\mathrm{Ext}^1_{\mathcal{O}}(L(y), L(x)) \neq 0$ *for some* $x, y \in W$, *then* $l(y) - l(x)$ *is odd.*

Proof. We may assume $x \leq y$, since the roles of x and y can be interchanged by dualizing. The short exact sequence

$$0 \to \mathrm{rad}\,\Delta(y) \to \Delta(y) \to L(y) \to 0$$

gives that $\mathrm{Ext}^1_{\mathcal{O}}(L(y), L(x)) \subseteq \mathrm{Ext}^1_{\mathcal{O}}(\Delta(y), L(x))$, because $\mathrm{Hom}_{\mathcal{O}}(\mathrm{rad}\,\Delta(y), L(x)) = 0$ when $x \leq y$. Now the statement follows from Proposition 7.4. □

Proposition 7.6. *Assume Claim 7.1. In the notation of Claim 7.2 we have for all* $y \in W$, $y \neq x$,

$$\mathrm{Hom}_{\mathcal{O}}(\Delta(y), U_s(sx)) \cong \begin{cases} \mathrm{Ext}^1_{\mathcal{O}}(\Delta(y), L(sx)) & \text{if } sy > y, l(y) \equiv l(x)(\mathrm{mod}\, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $U_s(sx)$ is s-finite by Theorem 6.3, the homomorphism space in question is trivial, if $L(y)$ is s-free. So, let us assume $sy > y$. Then we have $\Delta(sy) \subset \Delta(y)$. As $L(sy)$ is not s-finite we have $\mathrm{Hom}_{\mathcal{O}}(\Delta(sy), U_s(sx)) = 0$, i.e., $\mathrm{Hom}_{\mathcal{O}}(\Delta(y), U_s(sx)) \cong \mathrm{Hom}_{\mathcal{O}}\big(\Delta(y)/\Delta(sy), U_s(sx)\big)$. On the other hand, the four term sequence (2.4) gives the exact sequence

$$0 \to \Delta(y)/\Delta(sy) \to T_s\Delta(sy) \to \Delta(sy) \to 0.$$

We therefore have a resulting exact sequence

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{O}}\big(T_s\Delta(sy), T_sL(sx)\big) \to \mathrm{Hom}_{\mathcal{O}}\big(\Delta(y)/\Delta(sy), T_sL(sx)\big) \\ &\quad \to \mathrm{Ext}^1_{\mathcal{O}}\big(\Delta(sy), T_sL(sx)\big) \to \mathrm{Ext}^1_{\mathcal{O}}\big(T_s\Delta(sy), T_sL(sx)\big). \end{aligned}$$

Here the first term is zero because we can erase the two T_s (Corollary 4.2 using Theorem 2.2 and Theorem 6.1) and we assumed $y \neq x$. Also in the last term we can erase the T_s's and since $\Delta(sy) \cong T_s\Delta(y)$ by formula (2.3), we may similarly identify the third term with $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(y), L(sx))$. The proposition now follows by combining with Proposition 7.4. □

We shall need the following slight variation of a result due to Cline, Parshall and Scott.

Proposition 7.7 (cf. [CPS93, Theorem 4.1])**.** *Assume* $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(y), L(x)) = 0$ *for all* $x, y \in W$ *with* $l(y) \equiv l(x)(\mathrm{mod}\, 2)$. *Let* $U \in \mathcal{O}_0$ *with* $\mathrm{soc}\, U \cong \mathrm{hd}(U)$ *and with the property* $\mathrm{Hom}_{\mathcal{O}}(\Delta(y), U) = 0 = \mathrm{Hom}_{\mathcal{O}}(U, \nabla(y))$ *for all* $y \in W$ *with* $l(y)$ *odd (respectively even). Then* U *is completely reducible.*

Proof. Choose $y \in W$ such that $y \cdot 0$ is a maximal weight in U. We shall show that all occurrences of the composition factor $L(y)$ in U are in the head of U. The fact that the socle of U is isomorphic to its head then forces $L(y)$ to split off as a summand of U. By repeating this process we get the desired decomposition of U into a direct sum of simple modules.

So assume that $L(y)$ occurs in the radical $\mathrm{rad}(U)$ of U. Since $y \cdot 0$ is a maximal weight of $\mathrm{rad}(U)$, we have a nonzero homomorphism $\Delta(y) \to \mathrm{rad}(U)$. We let Y denote the image of this homomorphism and consider the map $Y \to L(y) \subset \nabla(y)$. The maximality of $y \cdot 0$ as a weight of U means that this extends to a homomorphism $U \to \nabla(y)$, because $\mathrm{Ext}^1_{\mathcal{O}}(U/Y, \nabla(y)) = 0$. The image Q of this homomorphism then contains $L(y)$ properly. So we may choose $z \in W$ such that $z \cdot 0$ is a maximal weight of $Q/L(y)$. Then we have

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\mathcal{O}}(\Delta(z), Q/L(y)) \quad &\subset \quad \mathrm{Hom}_{\mathcal{O}}(\Delta(z), \nabla(y)/L(y)) \\ &\cong \quad \mathrm{Ext}^1_{\mathcal{O}}(\Delta(z), L(y)). \end{aligned}$$

The last isomorphism comes from the fact that $\mathrm{Ext}^i_{\mathcal{O}}(\Delta(z), \nabla(y)) = 0$ for all i when $z \neq y$. By our assumption this implies that $l(z) - l(y)$ is odd. On the other hand, both $\mathrm{Hom}_{\mathcal{O}}(\Delta(y), U)$ and $\mathrm{Hom}_{\mathcal{O}}(U, \nabla(z))$, are nonzero (the latter because the composite $U \to Q \to Q/L(y) \to \nabla(z)$ is nonzero). Therefore $l(y)$ and $l(z)$ are both odd (respectively even) and we have a contradiction. □

The following result follows easily from what we have proved already:

Theorem 7.8. *Assume Claim 7.1. In the notation of Claim 7.2 we have that $U_s(sx)$ is completely reducible.*

Proof. Since $U_s(sx)$ is s-finite (Theorem 6.3), $\mathrm{Hom}_{\mathcal{O}}(U_s(sx), \nabla(y)) = 0$ if $y > sy$. The short exact sequence $U_s(sx) \hookrightarrow T_s L(sx) \twoheadrightarrow L(sx)$ gives rise to an exact sequence

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{O}}(L(sx), \nabla(y)) \to \mathrm{Hom}_{\mathcal{O}}(T_s L(sx), \nabla(y)) \to \mathrm{Hom}_{\mathcal{O}}(U_s(sx), \nabla(y)) \\ &\quad \to \mathrm{Ext}^1_{\mathcal{O}}(L(sx), \nabla(y)). \end{aligned}$$

Now let $\mathrm{Hom}_{\mathcal{O}}(U_s(sx), \nabla(y)) \neq 0$, hence $y < sy$ hence $\mathrm{Hom}_{\mathcal{O}}(T_s L(sx), \nabla(y)) \cong \mathrm{Hom}_{\mathcal{O}}(L(sx), \nabla(sy))$; i.e., $x = y$ or $0 \neq \mathrm{Ext}^1_{\mathcal{O}}(L(sx), \nabla(y)) = \mathrm{Ext}^1_{\mathcal{O}}(\Delta(y), L(sx))$. Proposition 7.4 implies $l(x) \equiv l(y) \bmod 2$. Together with Proposition 7.6 we get $\mathrm{Hom}_{\mathcal{O}}(\Delta(y), U_s(sx)) = 0 = \mathrm{Hom}_{\mathcal{O}}(U_s(sx), \nabla(y))$ if $l(x) \not\equiv l(y) \bmod 2$. Hence, Proposition 7.4 and Theorem 6.3 show that we may apply the previous Proposition 7.7. to $U = U_s(sx)$. □

The conclusion in the previous theorem is equivalent to Vogan's conjecture; see [Vog79] and [And86]. It implies the Kazhdan-Lusztig conjecture ([KL79, Conjecture 1.5]), namely that the following identity holds in the Grothendieck group of $\mathcal{O}_0$ for all x, $y \in W$:

$$[L(x)] = \sum_{y \in W} (-1)^{l(x)-l(y)} \; P_{yw_o, xw_o}(1) \; [\Delta(y)]. \tag{7.1}$$

Here, $P_{x,y} \in \mathbb{Z}[q]$ denotes the Kazhdan-Lusztig polynomial corresponding to x and y as defined in [KL79]. To see that (7.1) follows from Theorem 7.8 we observe first that for all $M \in \mathcal{O}_0$ we have the following equality:

$$[M] = \sum_{y \in W} \Big(\sum_{i \in \mathbb{N}} (-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}}(\Delta(y), M) \Big) [\Delta(y)]. \tag{7.2}$$

In fact, this is clear for $M = \nabla(x)$ with $x \in W$ arbitrary, because we have $\dim \operatorname{Ext}^i_{\mathcal{O}}(\Delta(y), \nabla(x)) = 1$ if $i = 0$ and $x = y$, and $\dim \operatorname{Ext}^i_{\mathcal{O}}(\Delta(y), \nabla(x)) = 0$ otherwise. The formula follows then for arbitrary M by additivity. To prove formula (7.1) it is therefore enough to prove the following

Proposition 7.9. *Assume Claim 7.1. Then we have for all x, $y \in W$,*

$$P_{yw_o,xw_o} = \sum_{y \in W} (-1)^i \dim \operatorname{Ext}^{l(y)-l(x)-2i}_{\mathcal{O}} \big(\Delta(y), L(x)\big)\, q^i.$$

Proof. The proof of this proposition is the same as that for Proposition 2.12 in [And86]. It goes by descending induction on the length of x and relies on the fact that if $sx > x$ for some $s \in S$, then $U_s(sx)$ is semisimple (Theorem 7.8). Propositions 7.4 and 7.6 are used also. □

Altogether, we proved that Claim 7.1 (or Claim 7.2) is analogous to the validity of the Kazhdan-Lusztig conjecture.

("Analogous" in the sense that for all $w_o \neq x \in W$ we have the non-vanishing of $\operatorname{Ext}^1_{\mathcal{O}}(L(x), L(sx))$ for some simple reflection s such that $sx > x$ instead of $\operatorname{Ext}^1_{\mathcal{O}}(L(x), L(xt)) \neq 0$ for some simple reflection t with $xt > x$. However, using the equivalence between $\mathcal{O}_0$ and a certain category of Harish-Chandra bimodules from [BG80] we get $\operatorname{Ext}^1_{\mathcal{O}}(L(x), L(sx)) = \operatorname{Ext}^1_{\mathcal{O}}(L(x^{-1}), L(x^{-1}s))$; in the bimodule picture it is just given by interchanging the left and right $\mathcal{U}$-module structure; cf. [Jan83, 7.28].)

A proof of Claim 7.1 or Claim 7.2 or a proof of any of the statements in Lemma 7.3 would therefore give a proof of the Kazhdan-Lusztig conjecture.

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