TQFT with corners and tilting functors in the Kac-Moody case

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Abstract

We study projective functors (i.e. direct summands of compositions of translations through walls) for parabolic versions of $\mathcal{O}$ as well as for integral regular blocks outside the critical hyperplanes in the symmetrizable Kac-Moody case. It turns out that in both situations the functors are completely determined by their restriction to the additive category generated by (the limit of) a 'full projective tilting' object. We describe how projective functors in the parabolic setup give rise to an invariant of tangle cobordisms and formulate a conjectural direct connection to Khovanov homology. Our main result, however, is the classification theorem for indecomposable projective functors in the Kac-Moody case verifying a conjecture of F. Malikov and I. Frenkel.

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Introduction

The original motivation of this paper was the spectacular "categorification program" described in [BFK99], where the authors propose a way to get tangle invariants via the representation theory of the Lie algebras $\mathfrak{sl}_n$. A first step in this programme was carried out in [Str05] where the main conjectures of [BFK99] were proved, thus providing a functorial tangle invariant. The functors there are built up from the so-called projective functors by some cone construction; these give tilting complexes and derived equivalences which provide a functorial action of the braid groups. The beautiful and amazing fact is that these functors lead to an "enriched" Jones invariant: the combinatorics of these functors (on the level of the Grothendieck group) is given by the Jones polynomial for tangles (see [Str05, Remark 7.3]). In the present paper we will mainly address the following three topics:

Invariants of tangles and cobordisms.

We follow the philosophy of [Rou] for example and consider not only the functors, which provide an invariant of tangles, but also natural transformations between them. This gives us (Theorem 2.5, Theorem 2.6) a "functorial realization" of the tangle 2-category: The functorial invariant from [Str05] associates to each tangle diagram a functor leading to a tangle invariant (up to shifts). In the present paper we will show that the indeterminacy up to shifts disappears if we work with oriented tangles instead and get therefore a functorial invariant of oriented tangles. Moreover, the tangle invariants will be enriched by assigning to each cobordism between two oriented tangles a natural transformation between the corresponding functors. We will prove that, up to scalars, this defines an invariant of cobordisms. Hence, we construct a 3-dimensional TQFT for manifolds with corner. It would be interesting to know whether the indeterminacy up to scalars can be explained in terms of an additional geometric or topological structure on the cobordisms.

Projective functors in the parabolic case and Khovanov homology.

We show that projective functors (and their morphisms) for the parabolic versions of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ are determined by evaluating at what we call a full projective tilting object (Theorem 1.8). This generalises the classification result of [BG80] for the projective functors of the category $\mathcal{O}$

to the parabolic versions of $\mathcal{O}$, as well as the result of W. Soergel [Soe90] which says that the combinatorics of the category $\mathcal{O}$ is given by the endomorphism ring of a full projective tilting module. We conjecture that this result provides a direct link between the functorial invariants from [Str05] and the Khovanov homology introduced in [Kho00] and [Kho02] (Conjecture 2.9 and Example 2.10). In particular, the combinatorics of Khovanov's homology should give a combinatorial description of the category of projective functors for certain parabolic categories $\mathcal{O}$ corresponding to $\mathfrak{sl}_n$. On the other hand the approach towards invariants of tangles and cobordisms we will describe is much "richer" than the one of [Kho02], since the underlying categories can be used to categorify (at least the main structures of) the tensor category of finite dimensional modules over the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. It provides for example a categorical interpretation of the (dual) canonical bases and the Schur-Weyl duality. The first steps in this direction (using a Koszul dual setup) can be found in [FKS], based on [BFK99].

The classification of projective functors in the Kac-Moody case:

The classification theorem of projective functors in [BG80] and the Kazhdan-Lusztig combinatorics together can be formulated (in "modern" language) as follows: *The projective endofunctors of the principal block of the category $\mathcal{O}$ associated with a semisimple complex Lie algebra $\mathfrak{g}$ form a ringoid which is isomorphic to the ringoid $\mathbb{Z}[W]$ of the associated Coxeter group.* We will generalise this result to the case where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. The first problem which occurs there is how to define projective functors. In [MF99a], [MF99b] such functors (so-called tilting functors) were defined for blocks of the category $\mathcal{O}$ outside the critical hyperplanes associated to a symmetrizable Kac-Moody algebra. The definition uses the highly non-trivial Kazhdan-Lusztig tensoring. In [MF99a] and [MF99b], I. Frenkel and F. Malikov also formulated a classification theorem in analogy to [BG80]. Unfortunately, there is a gap in the proof which, so far, cannot be fixed. Therefore, instead of following this path and using the Kazhdan-Lusztig tensoring, we define projective functors in terms of translation functors as they appear in [Nei88], [Nei89] and [Fie03], and then prove the classification theorem proposed by I. Frenkel and F. Malikov.

We begin the paper by recalling the basics about parabolic category $\mathcal{O}$. The first result is Theorem 1.8 which gives a description of homomorphism spaces between projective functors on the parabolic category $\mathcal{O}$. This result is probably well-known to specialists, but we were unable to find it in the literature. The theorem states that the morphisms between projective functors F and G are the bimodule homomorphisms between $F(T)$ and $G(T)$, where T is a full projective tilting module. In the special case of the principal block $\mathcal{O}_0$ of $\mathcal{O}$ this comes down to the well-known statement that morphisms between projective functors are simply the bimodule morphisms between certain special bimodules defined by W. Soergel. As an illustration we reprove the classification of projective functors from [BG80] using the Theorem 1.8 and Soergel's bimodules. Although the arguments are all taken from [Soe92], there is a slight (but for our purposes important) difference between our approach and the ones in [Soe92] and [BG80]: they work either with Harish-Chandra bimodules or with a deformed version of category $\mathcal{O}_0$ with deformation ring being the (localised) universal enveloping algebra $S(\mathfrak{h})$; we work with a specialisation with respect to the centre C of the category $\mathcal{O}_0$. For our purposes this is much stronger, since it can be generalised to the case of Kac-Moody algebras where it is not clear how to define Harish-Chandra bimodules. It fits also better with the combinatorics of $\mathcal{O}_0$ described in [Soe90] and generalised in [Fie04] to blocks (outside the critical hyperplanes) of category $\mathcal{O}$ for any symmetrizable Kac- Moody algebra.

This will be helpful in the last section when we define projective functors for blocks (outside the critical hyperplanes) of symmetrizable Kac–Moody algebras based on the translation functors defined in [Nei88], [Nei89] and [Fie03]. By definition, a projective functor is nothing else than a direct summand of direct sums of compositions of translations through walls. This definition works over any deformation ring, in particular we will deform with the centre Z of the category. Any projective functor F will have a deformed version F_Z.

For the principal block $\mathcal{O}_0$ of the classical category $\mathcal{O}$ of some semisimple Lie algebra $\mathfrak{g}$, the centre Z is the endomorphism ring of a full projective-tilting module T, namely the indecomposable projective module with anti-dominant highest weight ([Soe90]). On the other hand, Z is also the endomorphism ring of any Z-deformed Verma module $\Delta_Z(\lambda)$ in the principal block ([Soe92]). As a special case of Theorem 1.8 we get that any projective endofunctor F is completely determined by the $\mathfrak{g} \otimes_{\mathbb{C}} Z$-module $F(T)$ which is isomorphic (as a bimodule) to $F_Z\Delta_Z(0)$. The latter is a projective object in the Z-deformed category $\mathcal{O}$. If we specialise to the usual category $\mathcal{O}$ we get a projective object in the principal block of $\mathcal{O}$. This procedure gives a natural bijection between (indecomposable) projective endofunctors of $\mathcal{O}_0$ and (indecomposable) projective objects in $\mathcal{O}_0$ and a description of homomorphisms between such projective functors (see Section 1).

For Kac-Moody algebras the principal idea will be the same, but there are several obstacles to pass. Given a block $\mathcal{O}_\Lambda$ (outside the critical hyperplanes) it could happen that we have either (in the so-called positive level) enough projective modules, but no tilting modules and no anti-dominant weight, or we have tilting modules and an anti-dominant weight, but not necessarily projective modules (in the negative level). In particular, there need not to be a full projective-tilting module. We first concentrate on blocks, where there is an anti-dominant weight and use the idea of [RCW82] (in the special situation of [Fie03]) that one should use a limit of projective modules from truncated categories to get an analogue for Soergel's antidominant projective module and then construct a fake full projective tilting module. We show in Theorem 3.5 that its endomorphism ring coincides with the centre Z of the category $\mathcal{O}_\Lambda$ (verifying a conjecture of P. Fiebig). Applying the tilting equivalence τ from [Soe97a], which is based on independent work of S. Arkhipov and A. Voronov, one obtains some block $\mathcal{O}_{\tau(\Lambda)}$ which contains a dominant weight $\tau(\lambda)$. There, we have the deformed Verma module $\Delta_Z(\tau(\lambda))$ with endomorphism ring Z and have also translation functors (through walls) as defined in [Fie03]. In particular, our previous definition of projective endofunctors makes sense. Via the equivalence τ any such projective functor F induces some endofunctor, say $\tau(F)$, of the block we started with. We call these functors also projective. Via τ, the projective modules in blocks of positive level become tilting modules in blocks of negative level, hence these projective functors $\tau(F)$ map tilting modules to tilting modules. This is the reason why what we call "projective functors" are called "tilting functors" in [MF99b]. In analogy to the classical situation we will prove the following result which is a natural generalisation of [BG80] and coincides with the conjectural classification theorem in [MF99a] (note the typo in the formulation there).

Theorem. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbb{C}$. Let $\mathcal{O}_{\mathbb{C},\Lambda}$ be a regular block, outside the critical hyperplanes, of the corresponding category $\mathcal{O}$ such that Λ contains an antidominant weight λ.*

(a) Let Z denote the centre of $\mathcal{O}_{\mathbb{C},\Lambda}$. Let $F_{\mathbb{C}}, G_{\mathbb{C}} : \mathcal{O}_{\mathbb{C},\Lambda} \to \mathcal{O}_{\mathbb{C},\Lambda}$ be compositions of translations through walls with corresponding deformed functors $F_Z, G_Z : \mathcal{O}_{Z,\Lambda} \to \mathcal{O}_{Z,\Lambda}$. Let $\Delta_Z(\lambda)$ be the Z-deformed Verma module with highest weight λ. Then there is an isomorphism of vector spaces (or even of rings if $F_{\mathbb{C}} \cong G_{\mathbb{C}}$)

$$\operatorname{Hom}(F_{\mathbb{C}}, G_{\mathbb{C}}) \cong \operatorname{Hom}_{\mathfrak{g}\otimes Z}(F_Z\Delta_Z(\lambda), G_Z\Delta_Z(\lambda)).$$

(b) There are natural bijections of isomorphism classes

$$\begin{array}{cc}
\left\{\textit{indecomposable projective endofunctors of } \mathcal{O}_{\mathbb{C},\Lambda}\right\} & F \\
\updownarrow \textit{1:1} & \downarrow \\
\left\{\textit{indecomposable tilting objects of } \mathcal{O}_{\mathbb{C},\Lambda}\right\} & F(\Delta_{\mathbb{C}}(\lambda)) \\
\updownarrow \textit{1:1} & \downarrow \\
\left\{\textit{indecomposable projective objects of } \mathcal{O}_{\mathbb{C},\tau(\Lambda)}\right\} & \tau(F(\Delta_{\mathbb{C}}(\lambda))) \\
\updownarrow \textit{1:1} & \downarrow \\
\left\{\textit{indecomposable projective endofunctors of } \mathcal{O}_{\mathbb{C},\tau(\Lambda)}\right\} & \tau(F)
\end{array}$$

The theorem implies in the classical case of a finite dimensional Lie algebra the results of [Bac01, Section 4]. With the results from Kazhdan-Lusztig theory, it follows that *the projective functors from $\mathcal{O}_\Lambda$ to itself categorify the group algebra of the Coxeter group corresponding to the block $\mathcal{O}_\Lambda$.*

Certainly, one would like to have an explicit description of the centre Z. A first step to solve this problem was done in [Fie03], where the centre of a certain deformation $\mathcal{O}_{S_{(0)},\Lambda}$ of $\mathcal{O}_{\mathbb{C},\Lambda}$ was determined. However, it remained an essential open question if specialisation gives rise to the specialised centre (as was implicitly claimed in [Fie03]). The methods of [Fie03] are not sufficient to confirm this claim. In general, the deformed situation is quite different from and much "easier" than the specialised one (see e.g. [Fie04, Theorem 3.12] in comparison to Soergel's Structure Theorem [Soe90]). However, we prove in Theorem 3.5

that the centre behaves well under specialisation, and together with the main result of [Fie03] it turns out that the endomorphism ring of the fake full projective module mentioned above is a completion of the cohomology ring (see e.g. [KK79]) of the full flag manifold corresponding to the Langlands dual group. In particular, it has a natural grading. Therefore, the next step would be to consider a graded version of the category $\mathcal{O}$ for Kac–Moody algebras in analogy to [BGS96] and define graded versions of projective functors using the approach of [Str03a]. In this way one should get a categorification of the generic Hecke algebra instead of just a categorification of the group algebra.

Let us come back to our invariants of tangles and cobordisms mentioned earlier. A classical knot is an embedded circle in $\mathbb{R}^3$. Knot diagrams are generic projections of the image of such an embedding onto the plane. Two knot diagrams represent (up to ambient isotopy) the same knot if and only if they are related via a sequence of Reidemeister moves. These moves can be considered as surfaces in $\mathbb{R}^3 \times [0,1]$ properly mapped into $\mathbb{R}^2 \times [0,1]$, where the boundaries $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ are strings before and after the move. On the other hand, any such knot diagram can be decomposed into generating elementary parts by introducing a so-called height function. We have (see e.g. [CRS97, Theorem 2.2.1]) the following

Fact(R): two knot diagrams with a height function represent (up to ambient isotopy) the same knot if and only if they are related via a sequence of Reidemeister, T-, H- and N-moves. (see Section 2, Figure 2.3.1).

Instead of working with knots we will consider tangles. That means we fix a finite number (say m and n) of points in $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ respectively. An (m,n)-tangle is a set of disjoint smooth curves in $\mathbb{R}^2 \times [0,1]$, intersecting the boundary in $n+m$ points, which are exactly the $n+m$ endpoints of all the non-closed smooth curves. Any generic projection to the plane gives a tangle diagram. As for knot diagrams, Fact (R) holds and all the tangles can be written as a composition of elementary tangles which are depicted in Figure 2.2.1. Just as for knots, the moves have interpretations in terms of generic (knotted) surfaces (or cobordisms) properly embedded into $\mathbb{R}^3 \times [0,1]$, where the boundaries are the tangles before and after the move. Such a surface can be described via a family D_t, $t \in [0,1]$ of tangle diagrams by first projecting the surface to $\mathbb{R}^2 \times [0,1]$, and then letting D_t be the diagram representing the intersection with the planes $\mathbb{R}^2 \times \{t\}$. There are only finitely many critical points for t, which means between these points the diagrams undergo just planar isotopies. One can therefore describe the surfaces by a finite sequence of diagrams which represent the intersections at critical points. These sequences are called movies. In this way, any cobordism (or knotted surface in [CRS97]) between tangles can be described via a movie. In Figure 2.2.2 one can find 8 movies displayed. D. Roseman [Ros98] defined Reidemeister moves for surfaces and proved that two movies represent (up to ambient isotopy) the same knotted surface if and only if they are related via a sequence of Reidemeister moves.

In [CRS97] the authors gave a combinatorial description in form of a list of *elementary string interactions* and proved that the diagrams associated to any knotted surface is a finite sequence of elementary string interactions and that to each such finite sequence there exists in fact a corresponding knotted surface. Some elementary string interactions are depicted in Figure 2.2.2. The main theorem of [CRS97] gives a list of *movie moves* such that *two sequences of elementary string interactions represent (up to ambient isotopy) the same knotted surface if and only if they are related via a sequence of movie moves.* Examples of such movie moves are depicted in Figure 2.2.2.

In Theorem 2.6 we will extend the functorial invariant of tangles from [Str05] to an invariant of (oriented) tangles and cobordisms. To any tangle diagram we associate a functor and to each elementary string interaction we associate a natural transformation and show that these natural transformations satisfy (up to scalars) the relations from [CRS97]. Hence we get as a result a functorial invariant of oriented tangles and cobordisms which can be formulated as follows (using the notation from Section 1 and Section 2):

Theorem. *There is a 2-functor $\Phi^{or} : \mathcal{T}an^{or} \to \mathcal{F}unc$ such that*

1. *if t_1 and t_2 are 1-morphisms which differ by a sequence of Reidemeister, T-, H- or N-moves then there in isomorphism of functors $\Phi^{or}(t_1) \cong \Phi^{or}(t_2)$.*
2. *if c_1 and c_2 are sequences of generating 2-morphisms which differ by a sequence of movie moves then $\Phi^{or}(c_1) = \Phi^{or}(c_2)$.*

Invariants of tangles and cobordisms were already obtained by M. Jacobsson and M. Khovanov ([Jac04], [Kho06]). These two papers contain an "enrichment" of the knot and tangle invariant introduced by M. Khovanov ([Kho00], [Kho02]) who assigned to each tangle or link diagram the homology of a combinatorially defined complex of $\mathcal{H}_n$-bimodules for some explicit given algebra $\mathcal{H}_n$. (For a very nice overview with simplified arguments we refer to [Bar05]).

In the first sight, there is no connection between the homology introduced by M. Khovanov and the approach proposed in [BFK99] and [Str05]. However, it turns out that the arguments which establish the extension of [Str05] to an invariant of cobordisms just mimic the arguments given in [Kho06], provided one actually proves that the functors have similar properties to the functors given by tensoring with the complexes of $\mathcal{H}_n$-bimodules considered in [Kho02]. In Section 2 we will establish this and show that the functors share all the important nice properties. In [Bra02] it is already mentioned that a certain parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$ is equivalent to a module category over an algebra A such that the algebra $\mathcal{H}_n$ used in [Kho02] is a subquotient of A. The algebra A is described in terms of quivers and relations in [Bra02]. Built on the results of T. Braden, we will formulate a conjecture (Conjecture 2.9) which bridges the approaches of [BFK99] and [Kho00]. It says the following. Based on [BFK99], we associate in particular to each $(2m, 2n)$-tangle t a functor

$\Phi^{or}(t) : \bigoplus_{k=0}^{2n} \mathbf{D}^b(\text{gmof-}A_{2n}^k) \to \bigoplus_{k=0}^{2m} \mathbf{D}^b(\text{gmof-}A_{2m}^k)$, where $\mathbf{D}^b(\text{gmof-}A_n^k)$ denotes the bounded derived category of the graded version of $\mathcal{O}_0^{\mathfrak{p}_k}(\mathfrak{sl}_{2n})$ which is the principal block of the parabolic category $\mathcal{O}$ corresponding to $\mathfrak{sl}_{2n}$ with parabolic Weyl group isomorphic to $S_k \times S_{2n-k}$. Recall that for each $\mathcal{O}_0^{\mathfrak{p}_k}(\mathfrak{sl}_{2n})$ we have a full projective tilting module T_{2n}^k. If we first restrict $\Phi^{or}(t)$ to a functor from $\mathbf{D}^b(\text{gmof-}A_{2n}^n)$ to $\mathbf{D}^b(\text{gmof-}A_{2m}^k)$, and then to perfect complexes of projective tilting modules we finally assign to each $(2m, 2n)$-tangle a functor which can be realized as tensoring with a complex $\check{X}(\Phi^{or}(t))$ of $(\operatorname{End}_{\mathfrak{g}}(T_{2m}), \operatorname{End}_{\mathfrak{g}}(T_{2n}))$-bimodules. We conjecture the following direct connection to Khovanov homology:

Conjecture. 1. *For any natural number m, there is an isomorphism of algebras $p_m : \operatorname{End}_{\mathfrak{g}}(T_{2m}) \cong \mathcal{H}_m$, where $\mathcal{H}_m$ denotes Khovanov's algebra.*

2. *The homological tangle invariant $t \mapsto \mathbf{H}^\bullet(\check{X}_{\Phi^{or}(t)})$ is Khovanov's invariant.*

This conjecture will be illustrated for $n = m = 2$ in Example 2.10. If this conjecture is true, we get Khovanov's tangle homology as a special case of our much more general approach. Given such a direct connection would embed Khovanov's approach into the much richer structure coming from [BFK99], where most of the monoidal category of (quantised) $\mathfrak{sl}_2$-modules can be seen (see for example [FKS]), and vice versa it would provide the beautiful, computable and understandable combinatorics from Khovanov homology to describe the functorial tangle invariants.

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1 The finite dimensional parabolic situation

In this section we study the principal block of parabolic versions of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. We define the notion of projective functors generalising the definition in [BG80] and show that they are determined already by their restriction to projective modules which are also injective or tilting. The main result will be a theoretical description of morphism spaces between projective functors. It turns out that they are naturally graded. With the correct choice of a parabolic in type A, conjecturally the endomorphism ring of projective-injective modules gives rise to the algebra which defines Khovanov's homology and therefore provides a representation theoretic interpretation of Khovanov's combinatorial approach.

1.1 Notations and Preliminaries

For any ring R we denote by $\operatorname{mod}\text{-}R$ ($\operatorname{mof}\text{-}R$) and $R\text{-}\operatorname{mod}$ ($R\text{-}\operatorname{mof}$) respectively the category of (finitely generated) right/left R-modules. Likewise, $R\text{-}\operatorname{mof}\text{-}S = R\otimes S^{\mathrm{opp}}\text{-}\operatorname{mof} = \operatorname{mof}\text{-}R^{\mathrm{opp}}\otimes S$ denotes the category of finitely generated (R,S)-bimodules. If, additionally, R and S are graded rings the symbols $R\text{-}\operatorname{gmof}$, $\operatorname{gmof}\text{-}R$, $R\text{-}\operatorname{gmof}\text{-}S$ etc. denote the corresponding categories of *graded* modules with degree preserving morphisms. In the following *graded* always means $\mathbb{Z}$-graded. If $M = \oplus_{i\in\mathbb{Z}} M_i \in \operatorname{gmof}\text{-}R$ is a graded module and $n \in \mathbb{N}$, we denote by $M\langle n\rangle$ the object in $\operatorname{gmof}\text{-}R$ such that $M\langle n\rangle = M$ as right R-module, but the grading shifted by n, i.e. $(M\langle n\rangle)_i = M_{i-n}$. By an *algebra* we always mean a finite dimensional unitary associative algebra over the complex numbers.

Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra with its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We fix $\mathfrak{b} \supset \mathfrak{h}$, a Borel and a Cartan subalgebra. Let $\mathcal{U}(\mathfrak{b})$ and $\mathcal{U}(\mathfrak{h})$ be the corresponding universal enveloping algebras. Let $\mathcal{O} = \mathcal{O}(\mathfrak{g},\mathfrak{b})$ denote the corresponding highest weight category defined by Bernstein, Gelfand and Gelfand ([BGG76]). The objects of this category are finitely generated $\mathcal{U}(\mathfrak{g})$-modules M which are locally $\mathcal{U}(\mathfrak{b})$-finite and have a weight space decomposition $M = \oplus M_\lambda$, where $M_\lambda = \{m \in M \mid hm = \lambda(h)m\}$ for any $h \in \mathfrak{h}$. For precise definitions and properties of $\mathcal{O}$ we refer for example to [BG80], [Jan79], [Jan83].

Let W be the Weyl group corresponding to $\mathfrak{g}$ with longest element $w_0 \in W$ and let ρ denote the half-sum of positive roots. The category $\mathcal{O}$ has a decomposition into blocks, more precisely $\mathcal{O} = \bigoplus \mathcal{O}_{\lambda-\rho}$, where λ runs through the set of dominant weights and $\mathcal{O}_{\lambda-\rho}$ denotes the block containing the simple module with highest weight $\lambda-\rho$. In particular, $\mathcal{O}_{-\rho}$ is semisimple and $\mathcal{O}_0$ is the principal block containing the trivial representation.

We fix a parabolic subalgebra $\mathfrak{p}$ containing $\mathfrak{b}$ and consider the parabolic category $\mathcal{O}^{\mathfrak{p}}$ defined as the full subcategory of $\mathcal{O}$ given by locally $\mathfrak{p}$-finite objects. We refer to [RC80] and [Irv85] for properties. We **fix** $P^{\mathfrak{p}}$, a minimal projective generator of the main block $\mathcal{O}_0^{\mathfrak{p}}$ of $\mathcal{O}^{\mathfrak{p}}$ and denote $A^{\mathfrak{p}} = \operatorname{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$ its endomorphism ring. The block decomposition of $\mathcal{O}$ induces a decomposition $\mathcal{O}^{\mathfrak{p}} = \bigoplus \mathcal{O}^{\mathfrak{p}}_{\lambda-\rho}$. Note that the direct summands could become decomposable or even trivial. This is however not the case for the principal block $\mathcal{O}_0^{\mathfrak{p}}$ on which we will focus our attention.

1.2 The algebras $A^{\mathfrak{p}} = \operatorname{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$

Since $P^{\mathfrak{p}}$ is a projective generator, the functor $\operatorname{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}},\bullet)$ defines an equivalence of categories

$$\epsilon^{\mathfrak{p}} : \mathcal{O}_0^{\mathfrak{p}} \tilde{\to} \operatorname{mof}\text{-}A^{\mathfrak{p}}, \tag{1.1}$$

where mof-$A^{\mathfrak{p}}$ denotes the category of finitely generated right $A^{\mathfrak{p}}$-modules (see e.g. [Bas68, Section 2]).

Example 1.1. For $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{p}$ a maximal parabolic subalgebra with corresponding Weyl group isomorphic to $S_1 \times S_{n-1}$, the algebra $A^{\mathfrak{p}}$ is isomorphic to the path algebra of the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left, "a_1"] & \bullet \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \bullet \arrow[l, bend left, "b_2"] \arrow[r, dotted, no head] & \bullet \arrow[r, bend left, "a_{n-1}"] & \bullet \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

with vertices $1, 2, \ldots, n$ and the $2n$ arrows as indicated, with the relations $b_1 a_1 = 0$, $a_{i+1} a_i = 0 = b_i b_{i+1}$ and $a_i b_i = b_{i+1} a_{i+1}$ whenever the expression makes sense. This is exactly the algebra appearing in [KS02].

In general, a (handy) explicit description of the algebra $A^{\mathfrak{p}}$ is not known. However one can find partial information in the literature. Let us mention some: For type A or D and $\mathfrak{p}$ a maximal parabolic, the paper [Bra02] of T. Braden gives a very nice and useful, but unfortunately not very handy, description of the algebras $A^{\mathfrak{p}}$ in terms of quivers with relations. For the non-parabolic case $\mathfrak{p} = \mathfrak{b}$ some explicit examples can be found in [Str03b], obtained by an algorithm based on [Soe90] and recently generalised and substantially improved in [Vyb]. All the algebras $A^{\mathfrak{p}}$ can be equipped with a grading which turns them into Koszul algebras ([BGS96]). In the Example 1.1 the grading is obtained by putting every path in degree one. The dimension of the algebra can be determined using the combinatorics of parabolic Kazhdan-Lusztig polynomials (namely the $n_{x,y}$ in the notation of [Soe97b]; see e.g. [BGS96, Theorem 3.11.4]). The representation type of the algebras in question is determined in [BN05]. The categories $\mathcal{O}^{\mathfrak{p}}$ are highest weight categories (in the sense of [CPS88]), hence the algebras $A^{\mathfrak{p}}$ are quasi-hereditary in the sense of [Don98, Appendix].

1.3 $\mathcal{O}_0^{\mathfrak{p}}$ as highest weight category

For $\lambda \in \mathfrak{h}^*$ let $W_\lambda = \{w \in W \mid w \cdot \lambda\}$ denote the stabiliser of λ under the dot-action $w \cdot \lambda = w(\lambda + \rho) - \rho$. We denote by $W_{\mathfrak{p}} \subseteq W$ the parabolic subgroup corresponding to $\mathfrak{p}$. Let $W^{\mathfrak{p}}$ denote the set of minimal coset representatives in $W_{\mathfrak{p}} \backslash W$. The category $\mathcal{O}_0^{\mathfrak{p}}$ is a highest weight category where the *standard objects* are the (generalised) Verma modules $\Delta^{\mathfrak{p}}(x \cdot 0)$ with highest weights $x \cdot 0$ for $x \in W^{\mathfrak{p}}$. Recall that $\Delta^{\mathfrak{p}}(x \cdot 0)$ is the maximal quotient, contained in $\mathcal{O}^{\mathfrak{p}}$, of the Verma module $\Delta(x \cdot 0) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{x \cdot 0}$. We denote by $L(x \cdot 0) = L^{\mathfrak{p}}(x \cdot 0)$ the simple head of $\Delta^{\mathfrak{p}}(x \cdot 0)$ and by $P^{\mathfrak{p}}(x \cdot 0)$ its projective cover. (The latter is always considered as a direct summand of $P^{\mathfrak{p}}$.) By abuse of language, the standard, simple and projective modules in mof-$A^{\mathfrak{p}}$ are denoted by the same symbols. A module in $\mathcal{O}_0^{\mathfrak{p}}$ (or in mof-$A^{\mathfrak{p}}$ respectively) *has a standard flag*, if it has a filtration with subquotients isomorphic to generalised Verma modules (or standard objects). Any projective module has a standard flag.

1.4 The full projective tilting module

Let $T = T^{\mathfrak{p}} \in \text{mof-}A^{\mathfrak{p}}$ be a *full projective tilting* module, i.e. the direct sum over all modules constituting a system of representatives for the isomorphism classes of indecomposable objects which are at the same time projective, tilting and injective in $\text{mof-}A^{\mathfrak{p}}$. (Note that two of these three properties automatically force the third one, since there is a contravariant duality which preserves simple objects.) In [Irv85], Irving studied projective modules in $\mathcal{O}_0^{\mathfrak{p}}$ and found a very nice characterisation of projective modules which are also injective. We summarise the main results of his paper in the following proposition:

Proposition 1.2. *([Irv85]) Let $P \in \mathcal{O}_0^{\mathfrak{p}}$ be projective. Then the following are equivalent*

(i) P is injective in $\mathcal{O}_0^{\mathfrak{p}}$,

(ii) P is tilting in $\mathcal{O}_0^{\mathfrak{p}}$,

(iii) the head of P contains only simple modules of maximal possible Gelfand-Kirillov dimension,

(iv) any composition factor of the head of P occurs as a submodule in some standard module $\Delta^{\mathfrak{p}}(x \cdot 0)$,

(v) P is isomorphic to some direct sum of injective hulls of standard modules.

We choose T as a submodule of $P^{\mathfrak{p}}$ and its endomorphism ring $D^{\mathfrak{p}} = \mathrm{End}_{\mathfrak{g}}(T)$ as a subalgebra of $A^{\mathfrak{p}}$. Note that $D^{\mathfrak{p}}$ is always a finite dimensional Frobenius algebra. If we are in the special situation, where $\mathfrak{p} = \mathfrak{b}$ then T is the unique indecomposable projective-injective module (equivalent to the projective cover of the simple Verma module) with endomorphism ring $C = \mathcal{U}(\mathfrak{h})/(\mathcal{U}(\mathfrak{h})_+^W)$, the algebra of coinvariants ([Soe90]). In particular, $D^{\mathfrak{b}}$ is a commutative symmetric algebra.

Example 1.3. Let us consider the Example 1.1 and denote the indecomposable projective module corresponding to vertex i by $P(i)$.

a.) If $n = 3$ then the vertices 1, 2, 3 correspond to the simple objects $L(0)$, $L(s \cdot 0)$ and $L(st \cdot 0)$, where s, t are simple reflections. The socle series of the indecomposable projective modules are of the form $P^{\mathfrak{p}}(0) = L(0)|L(s \cdot 0)$, $P^{\mathfrak{p}}(s \cdot 0) = L(s \cdot 0)|L(0) \oplus L(st \cdot 0)|L(s \cdot 0)$ and $P^{\mathfrak{p}}(st \cdot 0) = L(st \cdot 0)|L(s \cdot 0)|L(st \cdot 0)$. Note that $P^{\mathfrak{p}}(0) = \Delta^{\mathfrak{p}}(0)$, $\Delta^{\mathfrak{p}}(s \cdot 0) = L(s \cdot 0)|L(st \cdot 0)$ and $\Delta^{\mathfrak{p}}(st \cdot 0) = L(st \cdot 0)$. We have $T = P(2) \oplus P(3)$. For arbitrary $n \geq 2$, we have $T = \bigoplus_{i=2}^{n} P(i)$. If $n = 2$ then $T = P(2)$ and its endomorphism algebra is the graded algebra $\mathbb{C}[x]/(x^2)$, where x is of degree 2. This is exactly the algebra $\mathcal{A}\langle 1 \rangle$ in [Kho00].

In general, the module T is quite difficult to describe and its endomorphism algebra is not known. It is known, however, that for $\mathfrak{g} = \mathfrak{sl}_n$ this algebra is symmetric, and depends only on the composition describing $\mathfrak{p}$, but not on the partition (see [MS]). Conjecturally the centre of this algebra is the cohomology ring of the associated Springer fibre (see [Kho04]).

1.5 The structure theorem for the category $\mathcal{O}_0^{\mathfrak{p}}$

The philosophy behind our approach and one reason why we want to consider full projective-tilting modules is that from the knowledge of the full-tilting module T with its endomorphism ring $\mathrm{End}_{\mathfrak{g}}(T)$ one could in principle recover the whole category $\mathcal{O}^{\mathfrak{p}}$. This point of view fits perfectly well with the special case of Soergel's description of $\mathcal{O}_0^{\mathfrak{b}}$ in terms of modules over the coinvariant algebra $\mathrm{End}_{\mathfrak{g}}(T^{\mathfrak{b}})$.

An important property of the full tilting module is given by the following proposition whose proof relies on the fact that projective objects in $\mathcal{O}_0^{\mathfrak{p}}$ can be built up from the projective Verma module using translation functors and makes clear how one should understand Irving's result from Proposition 1.2. It naturally generalises Soergel's structure theorem ([Soe90]). To formulate it we need a little bit more notation. Let $\theta_s : \mathcal{O}_0 \longrightarrow \mathcal{O}_0$ denote the *translation through* the s-wall as defined for example in [GJ81, Section 3]. The functor θ_s is exact and its own biadjoint, hence a so-called Frobenius functor. Note that θ_s maps the Verma module $\Delta(0)$ to the indecomposable projective module $P(s \cdot 0)$. By abuse of language we denote by the same symbol also its restriction $\theta_s : \mathcal{O}_0^{\mathfrak{p}} \longrightarrow \mathcal{O}_0^{\mathfrak{p}}$ as well as the induced endofunctor of $\mathrm{mof}\text{-}A^{\mathfrak{p}}$ (via (1.1)).

Definition 1.4. We call a functor $F : \mathcal{O}_0^{\mathfrak{p}} \longrightarrow \mathcal{O}_0^{\mathfrak{p}}$ (or $F : \mathrm{mof}\text{-}A^{\mathfrak{p}} \longrightarrow \mathrm{mof}\text{-}A^{\mathfrak{p}}$) *projective* if it is a direct sum of direct summands of some composition of translations through walls. In this case we also say F *is a projective functor on* $\mathcal{O}_0^{\mathfrak{p}}$ or $\mathrm{mof}\text{-}A^{\mathfrak{p}}$ respectively.

Remark 1.5. Note, that the classification theorem of projective functors from [BG80] (see Theorem 1.14 below) implies that our usage of the notation *projective functor* on $\mathcal{O}_0^{\mathfrak{b}}$ is compatible with the definition in [BG80]. In this classification the functor θ_s is then the (up to isomorphism unique) projective functor mapping $\Delta(0)$ to the projective module $P(s \cdot 0)$. Since $P(s \cdot 0)$ is indecomposable, so is the functor θ_s (for a direct proof see Lemma 1.10 below).

In the Example 1.3 (a), we have an inclusion of $P(0)$ into $P(s \cdot 0)$ such that the cokernel can be embedded into $P(st \cdot 0)$. In general the following holds:

Proposition 1.6. *Let $P \in \mathrm{gmof}\text{-}A^{\mathfrak{p}}$ be projective. Then there exists a projective-tilting-copresentation, that is an exact sequence of the form*

$$0 \to P \longrightarrow \bigoplus_I T \longrightarrow \bigoplus_J T,$$

for some finite index sets I, J.

Proof. By Proposition 1.2, a module from $\mathcal{O}_0^{\mathfrak{p}}$ is indecomposable projective-tilting if its socle (=head) is contained in the socle of a Verma module; and visa versa the projective hull of any simple composition factor occurring in the socle of a Verma module is injective. Therefore, any standard module, or even any module from $\mathrm{gmof}\text{-}A^{\mathfrak{p}}$ having a standard flag, embeds into a finite direct sum of copies of T. If $P = \Delta^{\mathfrak{p}}(0)$ is the projective standard module, then (by weight considerations) the cokernel of this embedding has again a standard filtration. Therefore, the statement of the lemma is true for $P = \Delta^p(0)$. Hence, there is a copresentation

$$0 \to \Delta^{\mathfrak{p}}(0) \xrightarrow{i} \bigoplus_I T \longrightarrow \bigoplus_J T,$$

for some finite sets I and J. Now, any projective module is of the form $F\Delta^{\mathfrak{p}}(0)$ for some projective functor F ([Irv85, Proposition (v)]). On the other hand, the socles of the indecomposable projective-injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ are exactly the simple objects with maximal Gelfand-Kirillov dimension (this is Proposition 1.2). Recall the following general fact: Assume $M \in \mathcal{O}_0^{\mathfrak{p}}$ and $\mathrm{Hom}_{\mathfrak{g}}(L, M) = 0$ for any simple object in $\mathcal{O}_0^{\mathfrak{p}}$ not having maximal Gelfand-Kirillov dimension and let $F : \mathcal{O}_0^{\mathfrak{p}} \to \mathcal{O}_0^{\mathfrak{p}}$ be a projective functor. Then, by definition, $FM \in \mathcal{O}_0^{\mathfrak{p}}$, but we also claim that $\mathrm{Hom}_{\mathfrak{g}}(L, FM) = 0$ for any simple object in $\mathcal{O}_0^{\mathfrak{p}}$ not having maximal Gelfand-Kirillov dimension. To show this let G be the adjoint functor of F. This is again a projective functor and we get $\mathrm{Hom}_{\mathfrak{g}}(L, FM) = \mathrm{Hom}(GL, M) = 0$, since G does not increase the Gelfand-Kirillov dimension ([Jan83, Lemma 8.8]), and therefore any quotient of GL has smaller Gelfand-Kirillov dimension than any arbitrary non-zero submodule of M ([Jan83, Lemma 8.6]). The claim follows.
In particular, the socle of the cokernel K of $F(i)$ has only composition factors of maximal Gelfand-Kirillov dimension. Applying again the Proposition 1.2 we know that the injective hull of K is also projective, hence a direct summand of some $\bigoplus_{I'} T$. On the other hand $F(T)$ is also projective and injective, hence a direct summand of some $\bigoplus_{I'} T$. The statement for general P follows. □

The following generalises [Soe90, Struktursatz] (see [Str03b, Theorem 10.1])

Corollary 1.7. *The functor* $\mathbb{V}^{\mathfrak{p}} = \mathrm{Hom}_{\mathrm{mof}\text{-}A^{\mathfrak{p}}}(T, \epsilon(\bullet)) : \mathcal{O}_0^{\mathfrak{p}} \to \mathrm{mof}\text{-}D^{\mathfrak{p}}$ *is fully faithful on projectives, i.e. it induces a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(P_1, P_2) \cong \mathrm{Hom}_{\mathrm{mof}\text{-}D^{\mathfrak{p}}}(\mathbb{V}^{\mathfrak{p}} P_1, \mathbb{V}^{\mathfrak{p}} P_2)$$

for projective objects $P_1,\ P_2 \in \mathcal{O}_0^{\mathfrak{p}}$.

Proof. Since any simple object occurring in the socle of a projective object in $\mathcal{O}_0^{\mathfrak{p}}$ is not annihilated by $\mathbb{V}^{\mathfrak{p}}$ (Proposition 1.2), the natural map is injective. It is obviously an isomorphism if $P_2 = T$, since the dimension on both sides is just the number of simple composition factors of P_1 of maximal Gelfand dimension which equals the dimension of $\mathbb{V}^{\mathfrak{p}} P_1$. Then it is also an isomorphism if P_2 is a finite direct sum of copies of T. For the general case, we take a copresentation $\mathcal{C}$ of P_2 coming from an injective-tilting copresentation for ϵP_2 via ϵ^{-1}. Then $\mathbb{V}^{\mathfrak{p}}\mathcal{C}$ is exact and stays exact when applying $\mathrm{Hom}_{\mathrm{mof}\text{-}D^{\mathfrak{p}}}(\mathbb{V}^{\mathfrak{p}} P_1, \bullet)$. On the other hand, applying $\mathrm{Hom}_{\mathfrak{g}}(P_1, \bullet)$ to $\mathcal{C}$ is also exact, since P_1 is projective. The desired result follows then using the Five lemma. □

1.6 Projective functors and projective tilting objects

Any projective functor from $\operatorname{mof}\text{-}A^{\mathfrak{p}}$ to $\operatorname{mof}\text{-}A^{\mathfrak{p}}$ preserves the additive category of projective tilting objects. In the following we want to show that the functor is already determined by its restriction to this additive category.

Let F be a projective functor from $\operatorname{mof}\text{-}A^{\mathfrak{p}}$ to $\operatorname{mof}\text{-}A^{\mathfrak{p}}$ then $F(T) \in \operatorname{mof}\text{-}A^{\mathfrak{p}}$ by definition. It has also a left $D^{\mathfrak{p}}$-module structure given by $d.t = F(d)(t)$ for any $d \in D^{\mathfrak{p}}$, $t \in F(T)$ giving rise to a $(D^{\mathfrak{p}}, A^{\mathfrak{p}})$-bimodule structure. By abstract nonsense ([Bas68, Section2]), since F is (right) exact, the functor F is isomorphic to tensoring with the $A^{\mathfrak{p}}$-bimodule $F(P^{\mathfrak{p}})$, however we claim the following stronger result describing the natural transformations $\operatorname{Hom}(F, G)$ between projective functors F and G:

Theorem 1.8. *Let F, G be projective functors on* $\operatorname{mof}\text{-}A^{\mathfrak{p}}$. *Let $T \in \operatorname{mof}\text{-}A^{\mathfrak{p}}$ be a full projective tilting module. There is an isomorphism of vector spaces (even of rings if $F = G$)*

$$\begin{array}{rcl} \operatorname{Hom}(F,G) & \tilde{\to} & \operatorname{Hom}_{D^{\mathfrak{p}}\text{-}\operatorname{mof}\text{-}A^{\mathfrak{p}}}(F(T), G(T)) \\ \phi & \mapsto & \phi_T. \end{array}$$

Proof. By naturality of the transformation ϕ, the map ϕ_T is in fact a $D^{\mathfrak{p}}$-$A^{\mathfrak{p}}$-bimodule morphism and therefore, our map is well-defined. We first show that it is injective. Assume $\phi_T = 0$. Let $P \in \operatorname{mof}\text{-}A^{\mathfrak{p}}$ be projective. There is an injective-tilting copresentation as in Lemma 1.6. Since F and G are exact and commute with finite direct sums we get a commutative diagram with exact rows of the form

$$\begin{tikzcd} 0 \arrow[r] & F(P) \arrow[r] \arrow[d, "\phi_P"] & F(\bigoplus T) \arrow[d, "0"] \\ 0 \arrow[r] & G(P) \arrow[r] & G(\bigoplus T). \end{tikzcd}$$

Since the vertical map on the right hand side is zero, we have $\phi_P = 0$ as well. Using a projective resolution we get $\phi_M = 0$ for any $M \in \mathcal{O}_0^{\mathfrak{p}}$. Therefore, the map $\phi \mapsto \phi_T$ is injective.
Let $\phi_T \in \operatorname{Hom}_{D^{\mathfrak{p}}\text{-}\operatorname{mof}\text{-}A^{\mathfrak{p}}}(F(T), G(T))$ and let $P \in \operatorname{mof}\text{-}A^{\mathfrak{p}}$ be projective with an injective-tilting copresentation

$$0 \to P \xrightarrow{f} \bigoplus_I T \xrightarrow{g} \bigoplus_J T,$$

Since F and G commute naturally with $\bigoplus$, we get a diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & F(P) \arrow[r, "F(f)"] & F(\bigoplus T) \arrow[r, "F(g)"] \arrow[d, "\wr"] & F(\bigoplus T) \arrow[d, "\wr"] \\
 & & \bigoplus F(T) \arrow[d, "\oplus\, \phi_T"] & \bigoplus F(T) \arrow[d, "\oplus\, \phi_T"] \\
 & & \bigoplus G(T) \arrow[d, "\wr"] & \bigoplus G(T) \arrow[d, "\wr"] \\
0 \arrow[r] & G(P) \arrow[r, "G(f)"] & G(\bigoplus T) \arrow[r, "G(g)"] & G(\bigoplus T),
\end{tikzcd}$$

where the rows are exact. The isomorphisms exist and are natural, since projective functors commute with direct sums (via the natural isomorphisms $(\oplus M) \otimes E \cong \oplus(M \otimes E)$ for $M \in \mathcal{O}_0^{\mathfrak{p}}$ and E a finite dimensional module). Since ϕ_T is a $(D^{\mathfrak{p}}, A^{\mathfrak{p}})$-bimodule morphism, it follows that the rectangle in the diagram above commutes. Since the rows are exact, restriction of the first vertical composition to $F(P)$ induces a unique morphism $\phi_P \in \operatorname{Hom}_{\operatorname{mof}\text{-}A^{\mathfrak{p}}}(F(P), G(P))$. Standard arguments show that ϕ_P does not depend on the chosen representation and defines in fact a natural transformation ϕ between F and G restricted to the category of projective right $A^{\mathfrak{p}}$-modules. For $N \in \operatorname{mof}\text{-}A^{\mathfrak{p}}$ arbitrary we choose a projective resolution $P^{\bullet}$. The naturality of ϕ defines a morphism of complexes $\Phi_N : F(P^{\bullet}) \longrightarrow G(P^{\bullet})$ inducing a unique map $\phi_N \in \operatorname{Hom}_{\operatorname{mof}\text{-}A^{\mathfrak{p}}}(F(N), G(N))$ by exactness of F and G. Again, standard arguments show that ϕ_N is independent of the chosen projective resolution. Moreover, this construction is natural in the sense that it defines a natural transformation between F and G as functors on $\operatorname{mof}\text{-}A^{\mathfrak{p}}$. □

Note that $\mathbb{V}^{\mathfrak{p}} F(T)$ is a $D^{\mathfrak{p}}$-bimodule, where $df = F(d) \circ f$ and $fd = f \circ d$ for $d \in D^{\mathfrak{p}}$, $f \in \mathbb{V}^{\mathfrak{p}} F(T)$. From Corollary 1.7 we directly get the following

Corollary 1.9. *With the assumptions of Theorem 1.8 there is an isomorphism of vector spaces*

$$\begin{array}{rcl} \operatorname{Hom}(F, G) & \tilde{\rightarrow} & \operatorname{Hom}_{D^{\mathfrak{p}}\text{-}\operatorname{mof}\text{-}D^{\mathfrak{p}}}(\mathbb{V}^{\mathfrak{p}} F(T), \mathbb{V}^{\mathfrak{p}} G(T)) \\ \phi & \mapsto & \mathbb{V}^{\mathfrak{p}}(\phi_T). \end{array}$$

1.7 The graded version

To make the result stronger we would like to work in a graded setup. From [BGS96], it is known that $A^{\mathfrak{p}}$ can be equipped with a non-negative grading turning it into a Koszul algebra. (In the example 1.1 the grading is given by putting all the arrows in degree 1).

We call a module $\tilde{P} \in \operatorname{gmof}\text{-}A^{\mathfrak{p}}$ *a graded lift* of $P \in \operatorname{mof}\text{-}A^{\mathfrak{p}}$ if it is isomorphic to P after forgetting the grading. Projective modules, standard objects and simple modules, for each of them exists a graded lift ([Str03a] or for a more general setup [Zhu04]). Moreover, these lifts are unique up to isomorphism and grading shift ([Str03a, Lemma 1.5]), since all these modules are indecomposable. We fix *standard lifts* with the property that their heads are concentrated in degree zero. In [Str03a], graded lifts of translation functors are defined. By a graded lift we mean the following:

Assume C, B are graded rings and let $F : C\text{-}\operatorname{mod} \to B\text{-}\operatorname{mod}$ be a functor. Then a functor $\tilde{F} : C\text{-}\operatorname{gmod} \to B\text{-}\operatorname{gmod}$ is a graded lift of F if it is a $\mathbb{Z}$-functor (i.e. it commutes with the grading shifts in the sense of [AJS94, E.3]), such that $\mathrm{f}_B\, \tilde{F} \cong F\, \mathrm{f}_C$, where $\mathrm{f}_C : \operatorname{gmof}\text{-}C \to \operatorname{mof}\text{-}C$, $\mathrm{f}_B : \operatorname{gmof}\text{-}B \to \operatorname{mof}\text{-}B$ denote the functors which forget the grading.

In [Str03a], it is shown that the translation functors $\theta_s : \operatorname{mof}\text{-}A^{\mathfrak{p}} \to \operatorname{mof}\text{-}A^{\mathfrak{p}}$, for any simple reflection s, have graded lifts

$$\tilde{\theta}_s : \operatorname{gmof}\text{-}A^{\mathfrak{b}} \to \operatorname{gmof} A^{\mathfrak{b}}.$$

On the other hand the natural projection of Koszul algebras $A \to A^{\mathfrak{p}}$ is graded and therefore, the functor $\tilde{\theta}_s$ restricts to a graded lift of $\theta_s : \operatorname{mof}\text{-}A^{\mathfrak{p}} \to \operatorname{mof}\text{-}A^{\mathfrak{p}}$ for any $\mathfrak{p}$. The following statement can be obtained as a direct consequence of the classification theorem of projective functors ([BGS96]). We give an easy direct proof. Recall that a functor F between abelian categories is *indecomposable* if $F \cong F_1 \oplus F_2$ implies $F_i = 0$ for at least one $i \in \{1, 2\}$.

Lemma 1.10. *For any simple reflection s, the functor $\theta_s : \operatorname{mof}\text{-}A^{\mathfrak{b}} \to \operatorname{mof} A^{\mathfrak{b}}$ is indecomposable. A graded lift $\tilde{\theta}_s$ is unique up to isomorphism and grading shift.*

Proof. Let $F = F_1 \oplus F_2$ such that $F_i \neq 0$ for $i = 1, 2$. In particular, $FM = F_1M \oplus F_2M$ for any standard module M. On the other hand, $\theta_s\Delta(x \cdot 0) \in \mathcal{O}_0$ is indecomposable. (From the properties of translation functors it follows easily that the socle of $\theta_s(M)$ is simple, hence θ_sM is indecomposable.) In particular, $F_{i(M)}(M) = 0$ for some $i(M) \in \{1, 2\}$. The socle of any standard module $\Delta(x \cdot 0) \in \mathcal{O}^{\mathfrak{b}}$ is of the form $\Delta(w_0 \cdot 0)$, hence not annihilated by any θ_s. From the exactness of F we get that $i := i(\Delta(w_0 \cdot 0)) = i(M)$ for any standard module M. Hence F_i is zero when restricted to the category of modules with standard flags. Since any projective module has a standard flag, the functor is zero on projectives, hence vanishes completely, since it is exact. Therefore, $F_i = 0$. This contradicts our assumption and therefore F is indecomposable. Since F is (right) exact, it is given by tensoring with some $A^{\mathfrak{b}}$-bimodule X. Since X is indecomposable, a graded lift is unique up to isomorphism and shift in the grading (this is [Str03a, Lemma 1.5] applied to $A^{\mathfrak{b}} \otimes (A^{\mathfrak{b}})^{opp}$). $\square$

Remark 1.11. Lemma 1.10 only holds in the case $\mathfrak{p} = \mathfrak{b}$. In general, the restriction of an indecomposable projective functor on $\mathcal{O}_0^{\mathfrak{b}}$ to a functor on $\mathcal{O}_0^{\mathfrak{p}}$ could be decomposable or even zero (see [Str05, Examples 3.7, Theorem 5.1]). In general it is not known how the restrictions decompose, not to mention a possible classification.

For any simple reflection s we fix a standard lift $\tilde{\theta}_s$ of θ_s : gmof-$A^{\mathfrak{p}} \to$ gmof-$A^{\mathfrak{p}}$ such that the standard lift of $\Delta(0)$ is mapped to the standard lift of $P(s \cdot 0)$. An endofunctor of gmof-$A^{\mathfrak{p}}$ is called *graded projective*, if it is a direct sum of (graded) direct summands of compositions of graded lifts of translation functors. If $M,\ N \in$ gmof-B for some graded ring B, then $\mathrm{Hom}_B(M,N)$ is graded by putting

$$\mathrm{Hom}_B(M,N)_n = \{f \in \mathrm{Hom}_B(M,N) \mid f(M_k) \subseteq M_{k+n}, \forall k \in \mathbb{Z}\}.$$

If C is also a graded ring and F : gmof-$B \to$ gmof-C is an exact functor then F is given by tensoring with some graded $B-C$-bimodule X_F. If G is another such functor we set

$$\mathrm{Hom}(F,G) \quad = \quad \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(F,Q)_n = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{B\otimes C^{\mathrm{opp}}\text{-}\mathrm{mof}}(X_F, X_G)_n.$$

We get the following refinement of Theorem 1.8:

Theorem 1.12. *Let $F,\ G$: gmof-$A^{\mathfrak{p}} \longrightarrow$ gmof-$A^{\mathfrak{p}}$ be graded projective functors. There is an isomorphism of graded vector spaces (even of rings if $F = G$)*

$$\begin{array}{rcl} \mathrm{Hom}(F,G) & \cong & \mathrm{Hom}_{D^{\mathfrak{p}}\text{-}\mathrm{mof}\text{-}A^{\mathfrak{p}}}(F(\tilde{T}), G(\tilde{T})) \\ \phi & \mapsto & \phi_{\tilde{T}}. \end{array}$$

Proof. Let $\Delta^{\mathfrak{p}}(0) \in$ gmof-$A^{\mathfrak{p}}$ be the standard lift of the projective standard object. There is an inclusion $\Delta^{\mathfrak{p}}(0)$ into $\bigoplus_{i\in I} \tilde{T}\langle i\rangle$ for a finite multiset I with elements from $\mathbb{Z}$, because the injective hull of $\Delta^{\mathfrak{p}}(0)$ is a direct sum of indecomposable projective-injective modules (compare the proof of Lemma 1.6). Since $\tilde{T}$ has a graded Verma flag ([Str03a, Theorem 7.2]), the cokernel of this inclusion has again a graded Verma flag by weight considerations. Therefore $P = \Delta^{\mathfrak{p}}(0)$ has an injective-projective resolution of graded right $A^{\mathfrak{p}}$-modules, i.e. there is an exact sequence of graded modules of the form

$$0 \to P \longrightarrow \bigoplus_{i\in I} \tilde{T}\langle i\rangle \longrightarrow \bigoplus_{j\in J} \tilde{T}\langle j\rangle \tag{1.2}$$

for some finite multisets I and J with entries in $\mathbb{Z}$. The statement follows then analogously to Proposition 1.6 and Theorem 1.8. $\square$

We denote by $\mathcal{Z}(R)$ the centre of any ring R. We want to give at least some description of the centre of $\mathcal{O}_0^{\mathfrak{p}} \cong$ gmof-$A^{\mathfrak{p}}$ which is by definition the centre of the ring $A^{\mathfrak{p}}$. Note that it inherits a grading from $A^{\mathfrak{p}}$.

Corollary 1.13. *Let* id *denote the identity functor on* gmof-$A^{\mathfrak{p}}$. *There are isomorphisms of (graded) rings*

$$\mathrm{End}(\mathrm{id}) \overset{(1)}{\cong} \mathrm{End}_{D^{\mathfrak{p}}\text{-}\mathrm{mof}\text{-}A^{\mathfrak{p}}}(\tilde{T}) \overset{(2)}{\cong} \mathcal{Z}(D) \overset{(3)}{\cong} \mathcal{Z}(A^{\mathfrak{p}}).$$

In particular, the homogeneous part of degree zero in $\mathcal{Z}(A^{\mathfrak{p}})$ has dimension one.

Proof. The existence of the first two isomorphism follows directly from the previous two theorems and Corollary 1.9. The isomorphism (3) is just obtained from the natural isomorphism $\mathrm{End}(\mathrm{id}) \cong \mathcal{Z}(A^{\mathfrak{p}})$ given by $\phi \mapsto \phi_{A^{\mathfrak{p}}}(1)$. The last statement follows directly from the definition of the grading on $A^{\mathfrak{p}}$, since $A^{\mathfrak{p}}$ is indecomposable and its homogeneous part of degree zero is semisimple. $\square$

Let $x \in W$ and $[x] = s_1 s_2 \cdot \ldots \cdot s_r$ be a fixed composition of simple reflections. We denote by $\theta_{[x]} = \theta_{s_r} \cdot \ldots \cdot \theta_{s_2}\theta_{s_1}$ the corresponding composition of translation functors and its graded version $\tilde{\theta}_{[x]} = \tilde{\theta}_{s_r} \cdots \tilde{\theta}_{s_2}\tilde{\theta}_{s_1}$, where $\tilde{\theta}_s$ denotes the standard graded lift of θ_s. Let R be a $\mathbb{C}$-algebra with a W-action. For $s \in W$ a simple reflection let R^s be the invariants under s. For $x \in W$ as above we denote $R_{[x]} = \bullet \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \otimes \cdots \otimes_{R^{s_r}} R$ considered as a functor on $\mathrm{mod}\text{-}R$. If additionally R is graded and the action of W is homogeneous, then $R_{[x]}$ is even an endofunctor of $\mathrm{gmod}\text{-}R$. In case $\mathcal{C}$ is any category having direct sums and $\mathcal{A}$ is a list of objects of $\mathcal{C}$ then we write $\mathrm{Ind}_{\mathcal{C}}(\mathcal{A})$ for the set of iso-classes of direct summands of elements in $\mathcal{A}$.

The next theorem is the classification theorem from [BG80]. We indicate a proof using Theorem 1.8, Corollary 1.9 and the deformation theory from [Soe92]:

Theorem 1.14. *Let $\mathfrak{p} = \mathfrak{b}$. There are natural bijections of isomorphism classes*

$$\begin{array}{cc}
\{\textit{indecomposable projective functors on}\,\mathrm{mof}\text{-}A^{\mathfrak{b}}\} & F \\
\updownarrow \textit{1:1} & \downarrow \\
\mathrm{Ind}_{D^{\mathfrak{b}}\text{-}\mathrm{mof}\text{-}A^{\mathfrak{b}}}(\theta_{[x]}T, x \in W) & F(T) \\
\updownarrow \textit{1:1} & \downarrow \\
\{\textit{indecomposable projective objects of}\ \mathrm{mof}\text{-}A^{\mathfrak{b}}\} & F(\Delta^{\mathfrak{b}}(0))
\end{array}$$

Proof. Theorem 1.8 implies that there is a bijection between indecomposable projective functors on $\mathrm{mof}\text{-}A^{\mathfrak{b}}$ and the set $\mathrm{Ind}_{D^{\mathfrak{b}}\text{-}\mathrm{mof}\text{-}A^{\mathfrak{p}}}(\theta_{[x]}T, x \in W)$, where T is the indecomposable projective-injective module in $\mathrm{mof}\text{-}A^{\mathfrak{b}}$. Consider the algebra $S = \mathcal{U}(\mathfrak{h})$ as a graded algebra with $S_2 = \mathfrak{h}$. Set $S_+ = (\mathfrak{h})$. By Soergel's Endomorphismensatz ([Soe90]) we have a canonical isomorphism $D^{\mathfrak{b}} \cong S/(S_+^W)$, the coinvariant algebra which we denote by C. By Corollary 1.9, the functor $\mathbb{V}^{\mathfrak{b}}$ defines a bijection between the isomorphism classes of indecomposable functors on $\mathrm{mod}\text{-}A^{\mathfrak{p}}$ and

$$\mathrm{Ind}_{C\text{-}\mathrm{mof}\text{-}D^{\mathfrak{b}}}(\mathbb{V}^{\mathfrak{b}}\theta_{[x]}T, x \in W) = \mathrm{Ind}_{C\text{-}\mathrm{mof}\text{-}C}(C_{[x]}(C), x \in W),$$

since it is known from [Soe90] that for any simple reflection s, there is an isomorphism of functors

$$\mathbb{V}^{\mathfrak{b}}\theta_s \cong \mathbb{V}^{\mathfrak{b}}(\bullet \otimes_{C^s} C). \tag{1.3}$$

There are isomorphisms of C-bimodules, or S-bimodules as follows:

$$C_{[x]}(C) = \big(S/(S_+^W)\big)_{[x]}(S/(S_+^W) = S_{[x]}(S/(S_+^W),$$

since $S^W \subseteq S^s$ for any simple reflection s, and $S_{[x]}(S)(S/(S^W_+) = S_{[x]}(S) \otimes_S (S/(S^W_+)$. Now, the S-bimodules $S_{[x]}(S)$ are exactly the *Soergel special bimodules* as defined in [Soe92] and [Soe05, Bemerkung 5.12]. By [Soe92, Proposition 11] specialisation gives a bijection between $\mathrm{Ind}_{S\text{-}\mathrm{mof}\text{-}S}(S_{[x]}(S), x \in W)$ and $\mathrm{Ind}_{C\text{-}\mathrm{mof}\text{-}C}(C_{[x]}(C), x \in W)$. The theorem follows therefore directly from [Soe05, Section 6]. $\square$

Remark 1.15. a.) Let $\mathfrak{p} = \mathfrak{b}$. Since all the constructions are compatible with gradings we get also a natural bijection of iso-classes of indecomposable graded projective endofunctors and indecomposable projective objects of $\mathrm{gmof}\text{-}A$.

b.) Theorem 1.14 is not true for arbitrary $\mathfrak{p}$ (see [Str05, Examples 3.7]).

c.) Using the results from Kazhdan-Lusztig theory one can view Theorem 1.14 as a categorification of the integral group algebra $\mathbb{Z}[W]$ as follows: There is an isomorphism of $\mathbb{Z}$-algebras between $\mathbb{Z}[W]$ and the Grothendieck ring of the indecomposable projective functors on $\mathcal{O}_0^{\mathfrak{b}}$ mapping the Kazhdan-Lusztig basis element C_w in the notation of [Soe97b] to the indecomposable projective functor which maps the projective Verma module in $\mathcal{O}_0^{\mathfrak{b}}$ to the indecomposable projective module with highest weight $w \cdot 0$. More generally, the graded versions of projective functors categorify the (generic) Hecke algebra corresponding to the Weyl group of $\mathfrak{g}$ (a precise formulation can be found in [Str05, Corollary 2.5]).

2 An invariant of tangle cobordisms

Now we would like to go one step further in the categorification program proposed in [BFK99]. The main result of [Str05] describes a functorial tangle invariant in terms of bounded derived categories of certain $\mathcal{O}_0^{\mathfrak{p}}$ for $\mathfrak{g}$ of type A. More precisely we associated to each (m, n)-tangle diagram a functor between a bounded derived category defined by certain $\mathcal{O}_0^{\mathfrak{p}}$ for $\mathfrak{sl}_m$ and a bounded derived category defined by certain $\mathcal{O}_0^{\mathfrak{p}}$ for $\mathfrak{sl}_n$. It was shown that, up to shifts, this assignment defines a functorial invariant. In this section we will prove what was already announced in [Str05], namely that this assignment can be extended firstly to a functorial invariant of oriented tangles (such that the discrepancy with respect to shifts disappears) and secondly to a 2-functor, which means one can associate to each oriented cobordism between two oriented tangles a (up to scalars) well-defined homomorphism between the corresponding functors. In this way the functorial tangle invariant is extended to an invariant of tangles with cobordisms.

2.1 The relevant functors and their morphisms

Let now be $\mathfrak{g} = \mathfrak{sl}_n$. We have $W = S_n$ generated by the simple reflections s_i, $1 \leq i < n$ with the relations $s_i s_j = s_j s_i$ if $|i - j| \geq 2$ and $s_i s_j s_i = s_j s_i s_j$ if

$|i-j|=1$. For $1 \leq k < n$ let $\mathfrak{p}_k$ denote the maximal parabolic subalgebra corresponding to the simple root α_k (i.e. the corresponding parabolic subgroup is generated by all s_j, where $j \neq k$). Set $\mathfrak{p}_0 = \mathfrak{p}_n = \mathfrak{g}$. To simplify notation let $\mathcal{O}_\mu^{\mathfrak{p}_i}$ denote the zero category if $n < i$ or $i < 0$ and $\mu \in \mathfrak{h}^*$.

Fix $1 \leq i < n$, $k \in \{0, \ldots, n\}$. Let $\theta_i = \theta_{s_i}$ denote the translation functor through the i-th wall with its standard lift $\tilde{\theta}_i$. Let $\lambda_i \in \mathfrak{h}^*$ be integral with stabiliser $W_{\lambda_i} = \{e, s_i\}$. In the following we need the translation functors

$$\theta_0^i : \qquad \mathcal{O}_0^{\mathfrak{p}_k}(\mathfrak{sl}_n) \longrightarrow \mathcal{O}_{\lambda_i}^{\mathfrak{p}_k}(\mathfrak{sl}_n) \tag{2.1}$$
$$\theta_i^0 : \qquad \mathcal{O}_{\lambda_i}^{\mathfrak{p}_k}(\mathfrak{sl}_n) \longrightarrow \mathcal{O}_0^{\mathfrak{p}_k}(\mathfrak{sl}_n) \tag{2.2}$$

on and *out off* the i-th wall. These are adjoint functors such that $\theta_i^0\theta_0^i \cong \theta_i$ (see e.g. [GJ81]). Enright and Shelton ([ES87, chapter 11]) defined an equivalence of categories

$$\zeta_{n,k} : \mathcal{O}_{\lambda_1}^{\mathfrak{p}_k}(\mathfrak{sl}_n) \tilde{\rightarrow} \mathcal{O}_0^{\mathfrak{p}_{k-1}}(\mathfrak{sl}_{n-2}). \tag{2.3}$$

We consider the following functors

$$\begin{aligned} \cap_{i,n}^k : \mathcal{O}_0^{\mathfrak{p}_k}(\mathfrak{sl}_n) &\quad\longrightarrow\quad \mathcal{O}_0^{\mathfrak{p}_{k-1}}(\mathfrak{sl}_{n-2}) \\ \cup_{i,n}^k : \mathcal{O}_0^{\mathfrak{p}_k}(\mathfrak{sl}_n) &\quad\longrightarrow\quad \mathcal{O}_0^{\mathfrak{p}_{k+1}}(\mathfrak{sl}_{n+2}) \end{aligned}$$

defined as

$$\begin{aligned} \cap_{i,n}^k &= \zeta_{n,k}\theta_0^1\theta_2\theta_3\cdots\theta_i \\ \cup_{i,n}^k &= \theta_i\theta_{i-1}\cdots\theta_2\theta_1^0\zeta_{n+2,k+1}^{-1} \end{aligned}$$

For $P_n^{\mathfrak{p}_k} \in \mathcal{O}^{\mathfrak{p}_k}(\mathfrak{sl}_n)$ a minimal projective generator we denote its endomorphism ring by A_n^k. For the following we fix equivalences (1.1) for any A_n^k and, concerning the notation, we will not distinguish the objects and functors on each side. We have graded versions $\tilde{\theta}_i$, $\tilde{\theta}_0^{\lambda_i}$, $\tilde{\theta}_{\lambda_i}^0$ of translation through, on and out of the wall as defined in [Str03a]. They are normalised such that $\tilde{\theta}_0^{\lambda_i}$ (resp. $\tilde{\theta}_{\lambda_i}^0$) maps a simple module concentrated in degree zero to a simple module (resp. a module with head) concentrated in degree -1. Since the equivalences $\zeta_{k,n}$ are compatible with the grading ([RH04]), we get also (standard) graded lifts $\tilde{\cap}_{i,n}^k$ and $\tilde{\cup}_{i,n}^k$ of $\cap_{i,n}^k$ and $\cup_{i,n}^k$ respectively. They have the following properties:

Lemma 2.1. *a.) The functors $\cap_{i,n}^k$ and $\cup_{i,n}^k$ are indecomposable. In particular, a graded lift is unique up to isomorphism and grading shift.*

b.) The standard lifts provide adjoint pairs of functors

$$\left(\tilde{\cap}_{i,n}^k\langle -1\rangle, \tilde{\cup}_{i,n-2}^{k-1}\right) \quad \textit{and} \quad \left(\tilde{\cup}_{i,n-2}^k\langle 1\rangle, \tilde{\cap}_{i,n}^{k+1}\right).$$

Proof. a.) Assume that $\cap_{i,n}^k$ decomposes as $\cap_{i,n}^k \cong F_1 \oplus F_2$. Then $\cup_{i,n-2}^{k-1}\cap_{i,n}^k \cong \cup_{i,n-2}^{k-1}F_1 \oplus \cup_{i,n-2}^{k-1}F_2$. Since $\cup_{i,n-2}^{k-1}\cap_{i,n}^k \cong \theta_i$ is indecomposable ([Str05, Lemma 6.3, Theorem 5.1]) it follows say $\cup_{i,n-2}^{k-1}F_1 = 0$. Assume $\cup_{i,n-2}^{k-1}(L) = 0$ for some simple object L. Then $0 = \theta_{i-1} \cup_{i,n-2}^{k-1} (L) \cong \theta_{i-1}\theta_i \cdots \theta_2\theta_1^0(L')$ for some simple object L'. Since $\theta_j\theta_{j'}\theta_j \cong \theta_j$ for $|j - j'| = 1$ ([Str05, Theorem 4.1]) it follows inductively that $\theta_2\theta_1^0(L') = 0$, hence $0 = \theta_0^1\theta_2\theta_1^0(L') \cong L'$ because of [BFK99, Lemma 4]. Therefore, $F_1 = 0$ contradicting our assumption. That means $\cap_{i,n}^k$ is indecomposable. The uniqueness of a graded lift follows form [Str03a, Lemma 1.5] (applied to the rings $A_n^k \otimes (A_{n-2}^{k-1})^{\mathrm{opp}}$ and $A_n^k \otimes (A_{n+2}^{k+1})^{\mathrm{opp}}$ respectively).
Assume $\cup_{i,n}^k$ decomposes as $\cup_{i,n}^k \cong F_1 \oplus F_2$ is decomposable. As above we deduce that, without loss of generality, $F_1\cap_{i,n+2}^{k+1} = 0$. Hence, $0 = F_1 \cap_{i,n+2}^{k+1} \cup_{i,n}^k \cong F_1 \oplus F_1$ by [Str05, Theorem 6.2]. We get $F_1 = 0$ contradicting our assumption.

b.) The adjointnesses follow directly from the definitions and the adjoint pairs $(\tilde{\theta}_1^0\langle 1\rangle, \tilde{\theta}_0^1)$, $(\tilde{\theta}_0^1\langle -1\rangle, \tilde{\theta}_1^0)$, and $(\tilde{\theta}_i, \tilde{\theta}_i)$ ([Str03a, Theorem 8.4, Corollary 8.3]). □

We describe the endomorphism rings of these functors.

Theorem 2.2. *For any $1 \leq i \leq n$ the following holds:*

a.) There is an isomorphism of graded vector spaces

$$\mathrm{End}(\tilde{\cup}_{i,n}^k) \cong \mathcal{Z}(A_n^k) \oplus \mathcal{Z}(A_n^k)\langle 2\rangle \cong \mathrm{End}(\tilde{\cap}_{i,n}^k).$$

Hence they are non-negatively graded and one-dimensional in degree zero.

b.) The vector spaces $\mathrm{Hom}(\mathrm{id}, \tilde{\theta}_i)$ and $\mathrm{Hom}(\tilde{\theta}_i, \mathrm{id})$ are both strictly positively graded and one-dimensional in degree one (with basis the adjunction morphism).

Proof. There are isomorphisms of graded vector spaces

$$\begin{array}{rcll} \mathrm{Hom}(\tilde{\cup}_{i,n}^k, \tilde{\cup}_{i,n}^k) & \cong & \mathrm{Hom}(\mathrm{id}\langle -1\rangle, \tilde{\cap}_{i,n+2}^{k+1}\tilde{\cup}_{i,n}^k) & \text{(Lemma 2.1)} \\ & \cong & \mathrm{Hom}(\mathrm{id}\langle -1\rangle, \mathrm{id}\langle -1\rangle \oplus \mathrm{id}\langle 1\rangle) & \text{([Str05, Theorem 6.2])} \\ & \cong & \mathrm{Hom}(\mathrm{id}, \mathrm{id} \oplus \mathrm{id}\langle 2\rangle). & \end{array}$$

The existence of the first isomorphism follows therefore from Corollary 1.13. Assuming the existence of the second isomorphism from part (a), part (b) follows by adjunction (Lemma 2.1) and the self-adjointness of $\tilde{\theta}_i$ ([Str03a, Corollary 6.3]), since $\tilde{\theta}_i \cong \cup_{i,n-2}^{k-1}\tilde{\cap}_{i,n}^k$ (see [Str05, Proposition 6.7]), and the adjunction morphisms are both of degree one ([Str03a]). To establish the second isomorphism of part (a) we have to work more. Let for the moment P be the chosen minimal projective generator of $\mathcal{O}_0^{\mathfrak{p}_k}(\mathfrak{sl}_n)$ with endomorphism ring $A = A_n^k$ and let P_λ be a minimal projective generator of $\mathcal{O}_\lambda^{\mathfrak{p}_k}$ where λ is an integral weight with stabiliser $W_\lambda = \{e, s_1\}$. Set $B = \mathrm{End}_\mathfrak{g}(P_\lambda)$. The composition $F = \theta_0^1\theta_2 \cdots \theta_i : \mathrm{mof}-A \to \mathrm{mof}-B$ is therefore given as tensoring with the bimodule

$$\mathbf{F} := \mathrm{Hom}_\mathfrak{g}(P_\lambda, F(P)) \in A\text{-}\mathrm{mof}-B$$

with the actions $af = F(a) \circ f$ and $fb = f \circ b$, where $f \in \mathbf{F}$, $a \in A$, $b \in B$. Let $\hat{F} = \theta_i \cdots \theta_2\theta_1^0$ be the adjoint functor with describing bimodule $\hat{\mathbf{F}} = \mathrm{Hom}_\mathfrak{g}(P, \hat{F}(P_\lambda)) \in B\text{-}\mathrm{mof}-A$. Since $F \in A\text{-}\mathrm{mof}$ and $B \in B\text{-}\mathrm{mof}$, the space $\mathrm{Hom}_{\mathrm{mof}-B}(\mathbf{F}, B)$ becomes naturally a (B, A)-bimodule. We claim that there are isomorphisms of (B, A)-bimodules

$$\mathrm{Hom}_{\mathrm{mof}-B}\left(\mathbf{F}, B\right) \xleftarrow{\Phi} \mathrm{Hom}_\mathfrak{g}\left(F(P), P_\lambda\right) \xrightarrow{\Psi} \hat{\mathbf{F}},$$

Here $(\Phi(g))(b) = g \circ b$ and Ψ is given by adjointness, i.e. $g \mapsto \hat{F}(g) \circ \eta$, where $\eta : \mathrm{id} \to \hat{F}F$ is the adjunction morphism. We calculate with $a \in A$ and $b \in B$ explicitly $\Psi(bga) = \hat{F}(bga) \circ \eta = \hat{F}(bg \circ F(a)) \circ \eta = \hat{F}(bg) \circ (\hat{F}F(a)) \circ \eta = \hat{F}(bg) \circ \eta \circ a = \hat{F}(b) \circ \Psi(g) \circ a = b\Psi(g)a$. Hence, Ψ is in fact an isomorphism of bimodules. Since $\Phi(bga)(h) = \Phi(b \circ g \circ F(a))(h) = \Phi(b \circ g)(F(a) \circ h) = (b\Phi(g))(F(a) \circ h) = (b\Phi(g)a)(h)$, the map Φ is compatible with the bimodule structures. It is obviously an isomorphism, since P_λ is a projective generator. Now, $\mathrm{Hom}_{\mathrm{mof}-B}(\bullet, B)$ defines an equivalence $e : A\text{-}\mathrm{mof}-B \tilde{\to} B\text{-}\mathrm{mof}-A$ with inverse functor $\mathrm{Hom}_{B\text{-}\mathrm{mof}}(\bullet, B)$. Therefore,

$$\mathrm{End}(F) \cong \mathrm{End}_{A\text{-}\mathrm{mof}-B}(\mathbf{F}) \cong \mathrm{End}_{B\text{-}\mathrm{mof}-A}(\hat{\mathbf{F}}) \cong \mathrm{End}(\hat{F}).$$

From the definitions of the graded translation functors it follows directly that all the isomorphisms are grading preserving. The Theorem follows. □

To simplify the setup, instead of working with derived categories, we will work with homotopy categories of complexes. To make it consistent with [Str05], one only has to replace all bounded derived categories appearing there by the bounded homotopy category of perfect complexes (for the general setup we refer for example to [KZ98]). Given a complex $(X^\bullet, d)$ we have the differentials $d : X^i \to X^{i-1}$ and $(X[k])^i = X^{i+k}$ for any $k \in \mathbb{Z}$.

Let $\mathbf{K}^b_{per}(\mathrm{gmof}-A^k_n)$ denote the bounded homotopy category of perfect complexes of graded A^k_n-modules. Let $\mathtt{C}^k_i$ be the endofunctor of $\mathbf{K}^b_{per}(\mathrm{gmof}-A^k_n)$ given by tensoring with the complex of graded A^k_n-bimodules

$$\cdots \to 0 \to A^k_n\langle 1\rangle \to \tilde{\theta}_i A^k_n \to 0 \to \cdots \tag{2.4}$$

where the map is the adjunction morphism (see Theorem 2.2) and $\tilde{\theta}_i A^k_n$ is concentrated in position zero. The functor $\mathtt{C}^k_i$ defines an auto-equivalence of $\mathbf{K}^b_{per}(\mathrm{gmof}-A^k_n)$ ([Str05, Section 7]). Let $\mathtt{K}^k_i$ be its inverse, that is the functor given by tensoring with the complex

$$\cdots \to 0 \to \tilde{\theta}_i A^k_n \to A^k_n\langle -1\rangle \to 0 \to \cdots \tag{2.5}$$

If F is a finite composition of functors of the form $\mathtt{C}_i^k$, $\mathtt{K}_i^k$ for some fixed k, then we have (via (2.4) and (2.5)) the corresponding complex, say $X(F)$ of graded A_n^k-bimodules. We consider $X(F)$ as an object in $\mathbf{K}^b(A_n^k\text{-}\operatorname{gmof}\text{-}A_n^k)$, the homotopy category of complexes of graded A_n^k-bimodules and denote by $\operatorname{End}(F)$ its endomorphism ring.

Lemma 2.3. *Let F be a finite composition of functors of the form $\mathtt{C}_i^k$, $\mathtt{K}_i^k$ for some fixed k.*

a.) There is an isomorphism of graded rings $\operatorname{End}(F) \cong \mathcal{Z}(A_n^k)$.

b.) If $G = \tilde{\cap}_{k,n}F$ and there exists a grading preserving isomorphism $f : G \cong H$ for some functor $H : \mathbf{K}^b_{per}(\operatorname{gmof} -A_n^k) \to \mathbf{K}^b_{per}(\operatorname{gmof} -A_{n-2}^{k-1})$. Then f is unique up to a scalar.

Proof. The first statement follows directly from [Ric89]. For a nice direct argument we refer to [Kho06, Proposition 1]. For the second statement we assume there is another isomorphism f', then $f'^{-1} \circ f \in \operatorname{End}(G)$ is of degree zero and by Theorem 2.2 a scalar multiple of the identity. □

We set $\mathtt{C}_i := \oplus_{k=0}^n \mathtt{C}_i^k$, considered as an auto-functor of $\bigoplus_{k=0}^n \mathbf{K}^b_{per}(\operatorname{gmof} -A_n^k)$ and let $\mathtt{K}_i = \oplus_{k=0}^n \mathtt{K}_i^k$ be its inverse.

2.2 The tangle 2-category and its generators

Let $A_n = \oplus_{k=1}^n A_n^k$. In [Str05], we assigned to a plane diagram of a tangle with m bottom and n top points a functor from $\mathbf{K}^b_{per}(\operatorname{gmof} -A_m)$ to $\mathbf{K}^b_{per}(\operatorname{gmof} -A_n)$. Up to shifts, this assignment provided an invariant of isomorphism classes of tangles. We also looked at the category $\mathcal{COB}$ of 2-cobordisms, where the objects are a finite number of labelled oriented one-manifolds and the morphisms are cobordisms. By considering the objects as special oriented $(0,0)$-tangles we assigned (in a 2-functorial way) to each object some functor and to each morphisms a natural transformation between the corresponding functors ([Str05, Theorem 8.1]) satisfying the defining relations for the isotopy classes of cobordisms. In the following we will show that this can be extended to arbitrary oriented tangles giving rise to a functorial invariant of tangles and cobordism.

More precisely, let $\mathcal{T}an$ denote the category of tangles, i.e. objects are the positive integers and morphisms are unframed tangle diagrams. Let $\mathcal{T}an^{or}$ be the 2-category of oriented tangles and cobordisms, i.e. objects are the positive integers, morphisms are unframed oriented tangles and 2-morphisms are diagrams of tangle cobordisms. For details see for example [BL03], [CRS97], [CS98], [Fis94]. This 2-category is of interest, since the 1-morphisms give rise to an algebraic description of tangles (and hence of knots and links), whereas the 2-morphisms describe compact surfaces smoothly embedded in $\mathbb{R}^4$.

The 1-morphisms are generated by the *elementary tangles* as depicted in Figure 2.2.1. i.e. the 1-morphisms are just the products of elementary tangle diagrams. For an (m,n)-tangle diagram T_1 and an (m',n')-tangle diagram T_2, the composition T_2T_1 is defined if and only if $n = m'$. In this case T_2T_1 is an

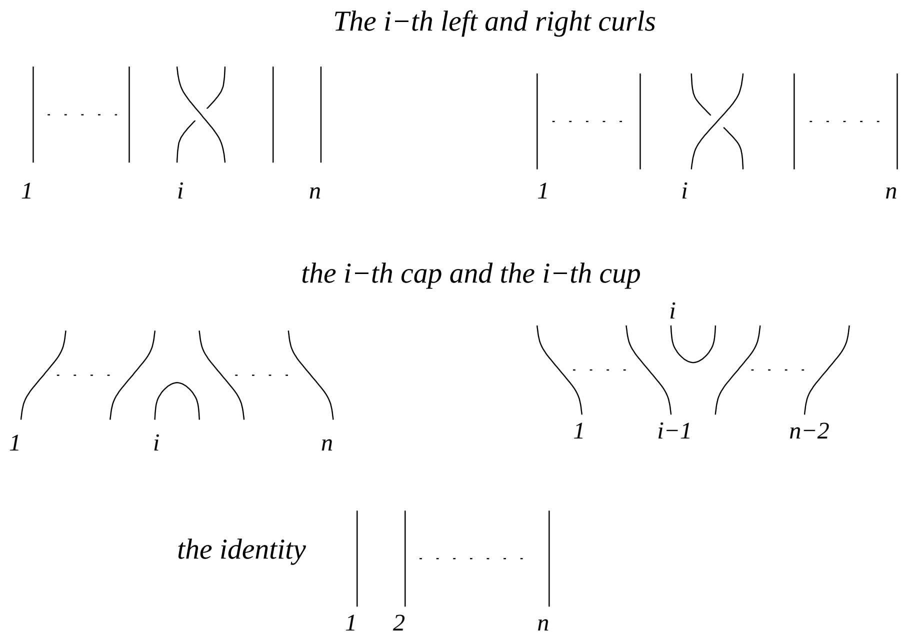


Figure 2.2.1: Elementary tangle diagrams generating the 1-morphisms

(m, n')-tangle diagram and obtained by putting T_2 on top of T_1 and identifying the bottom points of T_2 with the top points of T_1. The 2-morphisms in the category $\mathcal{T}an$ are diagrams of cobordisms generated by birth, death, saddle points, Reidemeister moves, shifting relative heights of distant crossings, local extrema, the identity morphisms, cusps on fold lines, and double point arcs crossing a fold line. The *typical generators* (apart from the identities) are depicted in Figure 2.3.1. These are the *elementary string interactions* from [CRS97], where one can also find the corresponding surfaces displayed. Any 2-morphism is a composition of generating 2-morphisms, the generating 2-morphisms are obtained by reading the typical generators either upwards or downwards, taking their vertical and horizontal mirror images and changing between negative and positive crossings. For details we again refer to [CRS97].

Recall that two tangle diagrams represent (up to ambient isotopy) the same tangle if they differ by a sequence of Reidemeister moves. As briefly mentioned in the introduction, to any cobordism (more precisely to any knotted surface in the sense of [CRS97]) there is an associated sequence of generating 2-morphisms. Moreover to any sequence of generating 2-morphisms there is a cobordism (knotted surface) whose diagram sequence is the given one ([CRS97, Theorem 3.5.4]). By [CRS97, Theorem 3.5.5], two sequences of generating 2-morphisms represent (up to ambient isotopy) the same cobordism if they differ by a sequence of so-called *movie moves* (see also [CRS97], [CS98], [BL03] or [Fis94]). Examples of movie moves are depicted in Figure 2.2.2. For a complete list of movie moves we refer to [CRS97, Theorem 3.5.5].

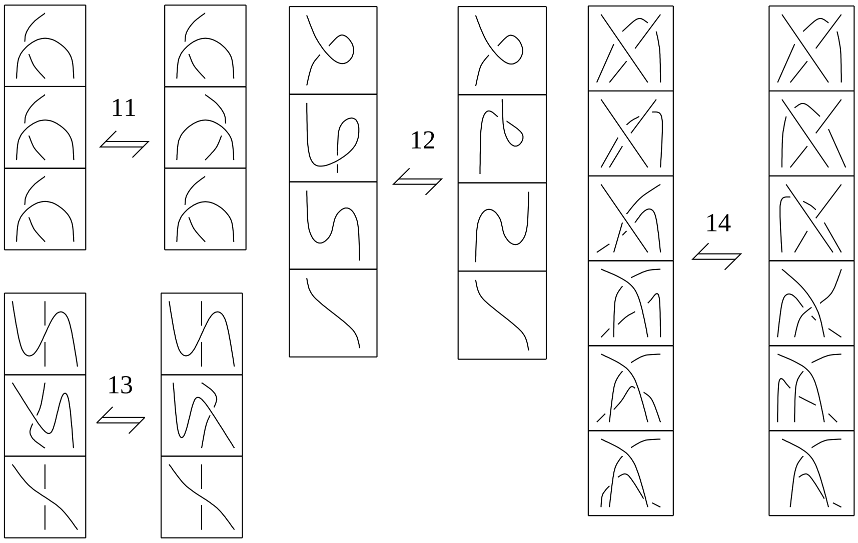
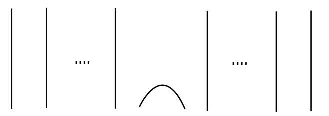


Figure 2.2.2: The movie moves 11 to 14

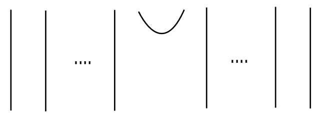
2.3 The functorial invariant of tangles

As suggested in [BFK99], we associated in [Str05] functors to elementary tangles as follows: To the identity tangle with n strands we associate the identity functor on $\mathbf{K}^b_{per}(\mathrm{gmof}\text{-}A_n) = \bigoplus_{k=0}^n \mathbf{K}^b_{per}(\mathrm{gmof}-A_n^k)$. For the U-turns we assign



1 2 i-1 i i+1 i+2 n-1 n

$$\rightsquigarrow \qquad \tilde{\cap}_{i,n}\langle 1\rangle : \mathbf{K}^b_{per}\big(\mathrm{gmof}\text{-}A_n\big) \longrightarrow \mathbf{K}^b_{per}\big(\mathrm{gmof}\text{-}A_{n-2}\big).$$



1 2 i-1 i n-1 n

$$\rightsquigarrow \qquad \tilde{\cup}_{i,n}\langle -1\rangle : \mathbf{K}^b_{per}\big(\mathrm{gmof}\text{-}A_n\big) \longrightarrow \mathbf{K}^b_{per}\big(\mathrm{gmof}\text{-}A_{n+2}\big).$$

To the i-th right twisted curl we associate the functor $\mathcal{C}_i := \mathtt{C}_i\langle 1\rangle$. To the i-th left twisted curl we associate the inverse functor $\mathcal{K}_i := \mathtt{K}_i\langle -1\rangle$.

Remark 2.4. The assignments differ slightly from the ones in [Str05] as follows: In [Str05] we took $\tilde{\cap}_{i,n}$ instead of $\tilde{\cap}_{i,n}\langle 1\rangle$ and also $\tilde{\cup}_{i,n}$ instead of $\tilde{\cap}_{i,n}\langle -1\rangle$.

Moreover, we swapped the assignments for left and right twisted curls. We introduced this renormalisation to make it compatible with Khovanov homology (see Conjecture 2.9).

Let $\mathcal{F}un$ denote the category which we define as follows: The objects are the bounded homotopy categories $\mathbf{K}^b_{per}(\text{gmof-}A_n)$ (i.e. the objects are indexed by the natural numbers). The 1-morphisms are functors between the corresponding categories.

Theorem 2.5. *(see [Str05, Theorem 7.1]) There is a functor*

$$\Phi : \mathcal{T}an \to \mathcal{F}un$$

which is given on objects by

$$n \in \mathbb{N} \mapsto \mathbf{K}^b_{per}(\text{gmof-}A_n),$$

and on elementary 1*-morphisms by the assignments above, such that if* t_1 *and* t_2 *are* 1*-morphisms which differ by a sequence of Reidemeister moves then there is an isomorphism of functors* $\Phi(t_1) \cong \Phi(t_2)\langle 3r\rangle[r]$ *for some* $r \in \mathbb{Z}$.

Proof. This is [Str05, Theorem 7.1] and Remark 2.4, since the renormalisation is compatible with the Reidemeister, N, H, and T-moves. ☐

Hence, the functor Φ defines, up to shifts, a functorial invariant of tangles. Moreover (see [Str05, Proof of Theorem 7.1]), it turns out that the shift problems only occur in H-moves.

2.4 Oriented tangles and cobordisms

Theorem 2.5 can be improved and made more natural by working with *oriented* tangles and cobordisms instead. Let $\mathcal{F}unc$ denote the 2-category with underlying category $\mathcal{F}un$: The objects are the bounded homotopy categories $\mathbf{K}^b_{per}(\text{gmof-}A_n)$, the 1-morphisms are functors between the corresponding categories, the 2-morphisms are the natural transformations between the functors, but after forgetting the grading and only up to a multiplication with a homogeneous element of degree 0 of the centre of the source or image category. We would like to construct a functorial invariant of oriented tangles and cobordisms.

To any elementary oriented tangle without crossing we associate the same functors as before. Let us consider the four H-moves of non-oriented tangles depicted in Figure 2.4.1, and let F_i $1 \leq i \leq 8$ be the corresponding functors. Then it is known that we have isomorphisms of $\mathbb{Z}$-functors $F_1\langle -3\rangle[-1] \cong F_2$, $F_3\langle -3\rangle[-1] \cong F_4$, $F_5\langle 3\rangle[1] \cong F_6$, $F_7\langle 3\rangle[1] \cong F_8$ (see [Str05, Proof of Theorem 7.1], note the signs appear because of Remark 2.4).

For oriented crossings we modify the assignment as follows: To an oriented crossing as depicted in Figure 2.4.2

we associate the functor $\mathcal{C}_i\langle -3\rangle[-1]$ and to an oriented crossing as depicted in Figure 2.4.3 we associate the functor $\mathcal{K}_i\langle 3\rangle[1]$. We get the oriented versions

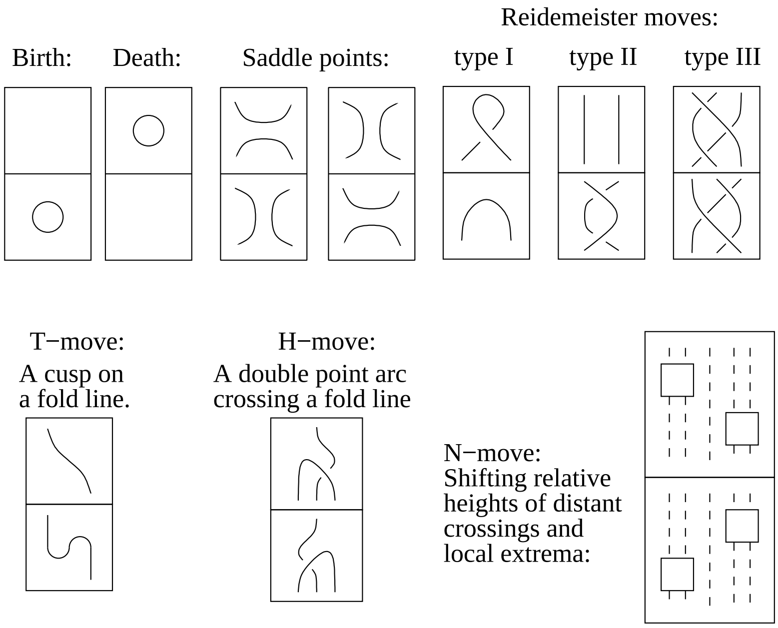


Figure 2.3.1: Typical generating 2-morphisms

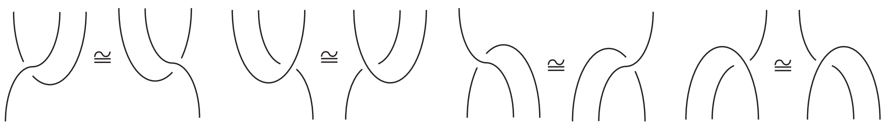


Figure 2.4.1: The H-moves

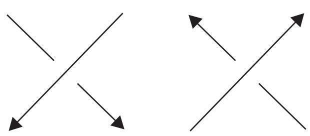


Figure 2.4.2: Oriented crossing inducing a shift $\langle -3\rangle[-1]$.

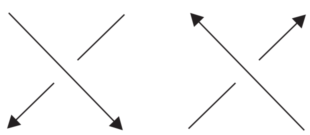


Figure 2.4.3: Oriented crossings inducing a shift $\langle 3\rangle[1]$.

of the H-moves as depicted in Figure 2.4.4. On can see immediately that our renormalisation precisely implies that the two functors corresponding to a move are now isomorphic (without any shifts!).

Theorem 2.6. *There is a functor of* 2*-categories*

$$\Phi^{or} : \mathcal{T}an^{or} \to \mathcal{F}unc$$

which is given on objects by

$$n \in \mathbb{N} \mapsto \mathbf{K}^b_{per}(\text{gmof -}A_n),$$

and on elementary 1*-morphisms by the assignments above, such that*

1. *if t_1 and t_2 are* 1*-morphisms which differ by a sequence of Reidemeister moves then there is an isomorphism of functors $\Phi^{or}(t_1) \cong \Phi^{or}(t_2)$.*
2. *if c_1 and c_2 are sequences of generating* 2*-morphisms which differ by a sequence of movie moves then $\Phi^{or}(c_1) = \Phi^{or}(c_2)$.*

The proof of this result requires the following auxiliary lemma:

Lemma 2.7. *Let n be any positive integer and let $1 \leq i, k \leq n$. There is an isomorphism of functors*

$$\phi_{i,n} : \tilde{\cap}^{\mathfrak{p}_k}_{i,n+2}\tilde{\cup}^{\mathfrak{p}_k}_{i,n} \cong \mathrm{id}\langle 1\rangle \oplus \mathrm{id}\langle 1\rangle : \text{gmof} -A^k_n \to \text{gmof} -A^k_n.$$

Proof. Since $\mathfrak{p}_k$ is a maximal parabolic, we have by [Str05, Theorem 4.1] isomorphisms of endofunctors of gmof $-A^k_n$ as follows

$$\tilde{\theta}_i\tilde{\theta}_j\tilde{\theta}_i \quad\cong\quad \tilde{\theta}_j\tilde{\theta}_i\tilde{\theta}_j \qquad\qquad \text{if } |i-j| = 1, \tag{2.6}$$

It is well-known (see e.g. [BG80]) that $\theta^i_0\theta^0_i \cong \mathrm{id} \oplus \mathrm{id}$ and $\tilde{\theta}^i_0\tilde{\theta}^0_i \cong \mathrm{id}\langle 1\rangle \oplus \mathrm{id}\langle -1\rangle$ (the latter by [Str03a, Theorem 8.2 (4)]). For $i = 1$, the statement of the lemma follows then directly from the definitions, since the Enright-Shelton equivalences are compatible with the grading [RH04]. For $i > 1$ we can deduce, by applying the relations (2.6), that it is enough to consider the case $i = 2$. Then, by [BFK99, Lemma 4], the lemma is true if we forget the grading. To determine the graded shifts we just apply [Str03a, Theorem 8.2 (2), Theorem 5.3] and the lemma follows. ☐

From now on we *fix* an isomorphism $\phi_{i,n}$ as in Lemma 2.7 for any i, n.

Proof of Theorem 2.6. From Theorem 2.5 we have the functor Φ. Since Reidemeister moves of type I do not involve any crossings of the form displayed in Figure 2.4.3 or Figure 2.4.2 we do not have to check anything there. For type II moves we are either in the situation as for tangles without orientation or we have the assignments from Figure 2.4.3 and Figure 2.4.2 and hence the statement follows. The moves of type III can be easily checked, since we know the result for the non-oriented case. For T-moves and N-moves nothing is to

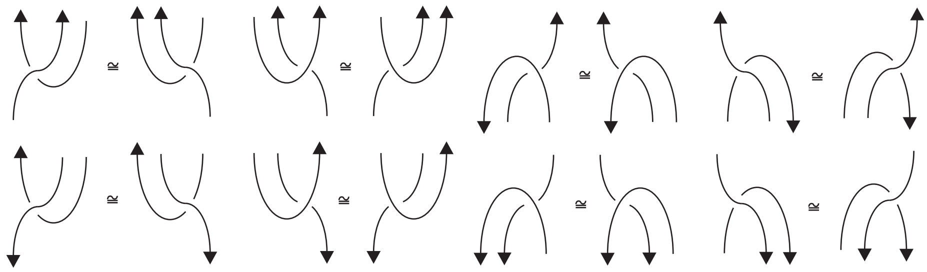


Figure 2.4.4: oriented H-moves

check. Finally, the functors are defined such that the H-moves work. Therefore Φ^{or} defines a functorial invariant of oriented tangles.

We have now to assign to each generating 2-morphism a functor between the corresponding functors.

Reidemeister moves: All different types of Reidemeister moves correspond to two isomorphic functors ([Str05, Proof of Theorem 7.1]). For any such move m we fix an isomorphism $i(m)$ which is grading preserving such that if we read the move upside down we get the inverse isomorphism. Then we define $\Phi(m) = i(m)$.

T-moves: By [Str05, Theorem 6.2, (6.1), (6.2)] we know that all the T-moves correspond to pairs of isomorphic functors if we forget the grading. Assume $\tilde{\cap}^{\mathfrak{p}_k}_{i+1,n+2}\tilde{\cup}^{\mathfrak{p}_k}_{i,n} \cong \mathrm{id}\langle k\rangle$ for some $k \in \mathbb{Z}$. Then we have $\tilde{\cup}^{\mathfrak{p}_k}_{i+1,n}\tilde{\cap}^{\mathfrak{p}_k}_{i+1,n+2}\tilde{\cup}^{\mathfrak{p}_k}_{i,n}\tilde{\cap}^{\mathfrak{p}_k}_{i,n+2} \cong \tilde{\cup}^{\mathfrak{p}_k}_{i+1,n}\tilde{\cap}^{\mathfrak{p}_k}_{i,n+2}\langle k\rangle$. However [Str05, Proposition 6.7] tells us that the left hand side of this isomorphism is isomorphic to $\tilde{\theta}_{i+1}\tilde{\theta}_i$, whereas the right hand side is isomorphic to $\tilde{\theta}_{i+1}\tilde{\theta}_i\langle k\rangle$. Hence $k = 0$. For any typical T-move m we fix a grading preserving isomorphism $t(m)$ such that if we read the move upside down we have the inverse isomorphism. Then we define $\Phi(m) = t(m)$.

H-move: All different types of H-moves correspond to two isomorphic functors. For any such move m we fix an isomorphism $h(m)$ which is grading preserving such that if we read the move upside down we get the inverse isomorphism. Then we define $\Phi(m) = h(m)$.

N-moves: All different types of N-moves correspond to two isomorphic functors. For any such move m we fix an isomorphism $n(m)$ which is grading preserving such that if we read the move upside down we get the inverse isomorphism. Then we define $\Phi(m) = n(m)$.

Saddle point: There are the degree preserving adjunction morphisms $\mathrm{id}\langle 1\rangle \to \tilde{\theta}_i$ and $\tilde{\theta}_i \to \mathrm{id}\langle -1\rangle$ (Theorem 2.2). We fix isomorphisms $\psi_{i,n} : \tilde{\cup}_{i,n}\tilde{\cap}_{i,n} \cong \tilde{\theta}_i$ which exist by [Str05, Proposition 6.7]. To each saddle point move we associate the natural transformation which is induced from the corresponding adjunction morphism.

Births/Deaths: Via the isomorphisms $\phi_{i,n}$ we get a surjection $\tilde{\cap}^{\mathfrak{p}_k}_{i,n+2}\tilde{\cup}^{\mathfrak{p}_k}_{i,n} \to \mathrm{id}\langle 1\rangle$ and an inclusion $\mathrm{id}\langle -1\rangle \to \tilde{\cap}^{\mathfrak{p}_k}_{i,n+2}\tilde{\cup}^{\mathfrak{p}_k}_{i,n}$ (both maps homogeneous of degree zero). To each birth move we associate the corresponding natural transformation (which is then homogeneous of degree -1).

It is left to show that the natural transformations satisfy the relations given by the movie moves. Because of our results in Section 2, the arguments are quite routine and mimic the arguments in [Kho06]. Now the statement follows directly by copying the arguments from [Kho06] if we make the following correspondences: Proposition 2 there corresponds to our Corollary 1.13; the Corollaries 1 and 2 to our Lemma 2.3 and Corollary 1.13. Khovanov's Proposition 3 with Corollary 3 should be replaced by Lemma 2.1 and Theorem 2.2. Corollary 4 corresponds to our Lemma 2.3 again. By repeating the arguments from [Kho06] the theorem follows. □

Remark 2.8. Although we have the invariant of the non-oriented tangles only up to shifts, the corresponding natural transformations would nevertheless satisfy the relations from [CRS97], since the only movie moves including H-moves are the moves 11 to 14 from Figure 2.2.2 and one can easily check that the corresponding natural transformations agree up to scalars, i.e. are well-defined.

2.5 A conjectural connection with Khovanov homology

In [Kho00] and [Kho02], M. Khovanov introduced a homological tangle and link invariant, now known as Khovanov homology. To any $(2n, 2m)$-tangle diagram he associated a certain complex of graded $(\mathcal{H}_n, \mathcal{H}_m)$-bimodules for some combinatorially defined algebras $\mathcal{H}_n$, $\mathcal{H}_m$. He proved that taking the graded cohomology groups defines an invariant of tangles and links. As already mention in [Bra02], there is a connection between the algebras $A^{\mathfrak{p}}$ from Section 1 where $\mathfrak{g} = \mathfrak{sl}_{2n}$ and $\mathfrak{p}$ is the parabolic subalgebra given by all (n, n)-upper block matrices and Khovanov's algebra $\mathcal{H}_n$.

On the other hand, to each $(2n, 2m)$-tangle diagram t we associated a functor $F = \Phi^{or}(t) : \mathbf{K}^b_{per}\big(\text{gmof-}A_{2n}\big) \longrightarrow \mathbf{K}^b_{per}\big(\text{gmof-}A_{2m}\big)$ which can be described by tensoring with a complex X_F of graded (A_{2m}, A_{2n})-bimodules. Let F' denote the restriction of F to a functor $F' : \mathbf{K}^b_{per}\big(\text{gmof-}A^n_{2n}\big) \longrightarrow \mathbf{K}^b_{per}\big(\text{gmof-}A^m_{2m}\big)$ and let X'_F be the associated complex of bimodules. From Theorem 1.8 we know that we do not loose any information if we restrict the functors to the categories of projective-tilting modules. Let $\check{X}_F$ denote the corresponding complex of graded (D^m_{2m}, D^n_{2n})-bimodules. Let $\mathbf{H}^\bullet(F)$ denote the graded cohomology of $\check{X}_F$.

The following conjecture relates the functorial invariant with Khovanov's invariant:

Conjecture 2.9. *The homological tangle invariant $t \mapsto \mathbf{H}^\bullet(\check{X}_{\Phi^{or}(t)})$ is Khovanov's invariant.*

We would like to illustrate this on an

Example 2.10. Let us consider the 2-category of oriented $(2, 2)$-tangles. It has a single object, 2, and the functorial invariant Φ^{or} from Theorem 2.6 assigns to each oriented $(2, 2)$ tangle t an endofunctor of the bounded derived category of mof-A_2 and, via restriction, of mof-A_2^1. Recall that mof-$A_2^1 \cong \mathcal{O}_0(\mathfrak{sl}_2)$. From Example 1.3 we have $D_2^2 \cong \mathbb{C}[x]/(x^2) = \mathcal{A}\langle 1\rangle$. Set $C = D_2^2$. Let us describe $\check{X}_{\Phi^{or}(t)}$ explicitly in terms of graded C-bimodules. To the flat diagram with two vertical strands we associated the identity functor which is given by tensoring with the graded (C,C)-bimodule C. To the other flat (cap-cup) diagram we associated the translation functor $\tilde{\theta}_1$. Via the functor $\mathbb{V}^{\mathfrak{b}}$ from Corollary 1.7 this becomes tensoring with $C \otimes_{\mathbb{C}} C\langle -1\rangle$ (see [Soe90]). To the left twisted curls depicted in Figure 2.4.3 we associated the functor given by tensoring with the complex of graded C-bimodules

$$\left(\cdots \to 0 \to C \otimes_{\mathbb{C}} C\langle -1\rangle \xrightarrow{m} C\langle -1\rangle \to 0 \cdots\right)\langle -1\rangle\langle 3\rangle[1],$$

which is

$$\cdots \to 0 \to C \otimes_{\mathbb{C}} C\langle 1\rangle \xrightarrow{m} C\langle 1\rangle \to 0 \to \cdots, \tag{2.7}$$

where m is the multiplication map and $C\langle 1\rangle$ is concentrated in position zero of the complex (see also [Str05, Section 8]). To the right twisted curl depicted in Figure 2.4.3 we associated the functor given by tensoring with the complex

$$\left(\cdots \to 0 \to C\langle 1\rangle \xrightarrow{\Delta} C \otimes_{\mathbb{C}} C\langle -1\rangle \to 0 \cdots\right)\langle 1\rangle\langle -3\rangle[-1],$$

where Δ is mapping 1 to $X \otimes 1 + X \otimes 1$. Hence we get the complex

$$\cdots \to 0 \to C\langle -1\rangle \xrightarrow{\Delta} C \otimes_{\mathbb{C}} C\langle -3\rangle \to 0 \cdots, \tag{2.8}$$

where $C\langle -1\rangle$ is concentrated in position zero of the complex.

On the other hand Khovanov associated to the flat diagram with two vertical strands the algebra $\mathcal{A}\langle 1\rangle = C$ and to the other flat tangle the C-bimodule $\mathcal{A} \otimes \mathcal{A}\langle 1\rangle \cong C \otimes_{\mathbb{C}} C\langle -1\rangle$. To the left twisted curls depicted in Figure 2.4.3 corresponds the complex

$$\begin{array}{rl} & \left(\cdots \to 0 \to \quad \mathcal{A} \otimes \mathcal{A}\langle 1\rangle \quad \xrightarrow{m} \quad \mathcal{A}\langle 1-1\rangle \to \quad 0 \to \cdots\right)\langle 2\rangle[-1], \\ = & \left(\cdots \to 0 \to \quad \mathcal{A} \otimes \mathcal{A}\langle 3\rangle \xrightarrow{m} \quad \mathcal{A}\langle 2\rangle \quad \to \quad 0 \to \cdots\right) \end{array}$$

where $\mathcal{A}\langle 2\rangle \cong C\langle 1\rangle$ is concentrated in position zero. Hence the complex coincides with (2.7). The complex associated in [Kho02] to the right twisted curl depicted in Figure 2.4.2 is exactly the complex (2.8).

Therefore, in the example of $(2,2)$-tangles, Khovanov homology is nothing else than the homology of the functorial invariant from 2.6 restricted to the underlying category given by projective-tilting modules.

3 Kac-Moody algebras

In this section we consider projective (respectively tilting) functors for symmetrizable Kac-Moody algebras. Given any regular block outside the critical hyperplanes, the main result is the classification theorem of these projective endofunctors - which can be viewed as a categorification of the group algebra of the integral Weyl group associated to the block. The result was conjectured in [MF99b], where the authors defined tilting functors via the so-called Kazhdan-Lusztig tensoring. Since this construction is highly non-trivial we decided to work with an alternative definition instead. Our approach is based on the translation functors introduced and described in [Nei88], [Nei89] and [Fie03].

Let now $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra (in the sense of [Kac90]) over the complex numbers with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{b}$. Let $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, $B = \mathcal{U}(\mathfrak{b})$ and $S = \mathcal{U}(\mathfrak{h})$ denote the corresponding universal enveloping algebras. Let $\mathcal{W}$ be the Weyl group. Let T be a *deformation algebra*, i.e an associative commutative noetherian unital S-algebra. Important examples will be $T = \mathbb{C} \cong S/(\mathfrak{h})$ and $T = S_{(0)}$ which is the localisation of S at $(\mathfrak{h})$. Let $\mathrm{t} : S \to T$ be the morphism defining the S-structure on T.
Since $\mathfrak{g}$ is symmetrizable we have a non-degenerated symmetric bilinear form on $\mathfrak{g}$ inducing a non-degenerated bilinear form $(\bullet, \bullet) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ on $\mathfrak{h}$ via restriction and dualising. This extends T-linearly to a bilinear form $(\bullet, \bullet)_T : \mathfrak{h}^* \otimes T \times \mathfrak{h}^* \otimes T \to T$. We consider the restriction $\mathrm{t}_\mathfrak{h}$ of t to $\mathfrak{h}$ as an element of $\mathrm{Hom}_\mathbb{C}(\mathfrak{h}, T) = \mathfrak{h}^* \otimes T$ and define $h_\mu = (\mu, \mathrm{t}_\mathfrak{h})_T$ for any $\mu \in \mathfrak{h}^* \subset \mathfrak{h}^* \otimes T$. Let $(\bullet, \bullet)_\mathbb{C}$ be the specialisation of $(\bullet, \bullet)_T$. A weight $\lambda \in \mathfrak{h}^*$ lies *outside the critical hyperplanes* if $(\lambda + \rho + \mathrm{t}_\mathfrak{h}, \beta)_\mathbb{C} \notin \mathbb{Z}(\beta, \beta)_\mathbb{C}$ for any imaginary root β. We fix $\rho \in \mathfrak{h}^*$ such that $(\rho, \alpha) = 1$ for any simple root α. The dot-action of the Weyl group on $\mathfrak{h}^*$ is defined as in the finite dimensional case. In the following we assume the reader to be familiar with the definition and results of [Fie03] and [Fie04].

3.1 The deformed category $\mathcal{O}_T$

We consider the *T-deformed category* $\mathcal{O}$ denoted by $\mathcal{O}_T$. It is the full subcategory of the category of $\mathcal{U} \otimes T$-modules given by locally $B \otimes T$-finite objects having a weight decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, such that

$$M_\lambda = \{m \in M \mid hm = (\lambda + \mathrm{t})(h)m, \text{ for all } h \in \mathfrak{h}\},$$

where the $(\lambda + \mathrm{t})(h)$ are considered as elements of T, see e.g. [Fie03, Section 2.1]. Set $\mathfrak{g}_T = \mathfrak{g} \otimes T$. For $\lambda \in \mathfrak{h}^*$ the composition $B \to S \xrightarrow{\lambda + \mathrm{t}} T$ defines a B-structure on T which commutes with the usual left T-action. The resulting $B \otimes T$-structure on T will be denoted by T_λ. We denote by $\Delta_T(\lambda) \in \mathcal{O}_T$ the *T-deformed Verma module* $\Delta_T(\lambda) = \mathcal{U} \otimes_B T_\lambda$ with highest weight λ. If T is local with maximal ideal $\mathfrak{m}$ or $T = \mathbb{C}$ then the isomorphism classes $L_T(\lambda)$ of simple objects in $\mathcal{O}_T$ are parametrised by elements $\lambda \in \mathfrak{h}^*$ by taking their highest weights. Under the specialisation functor $T/\mathfrak{m} \otimes_T \bullet$ the simple objects in $\mathcal{O}_T$ become the simple objects in the ordinary BGG-category $\mathcal{O}_{T/\mathfrak{m}}$ (where the modules need not to be finitely generated). For details we refer to [Fie03, Proposition 2.1].

3.2 Blocks outside the critical hyperplanes

Let now T be a *local* deformation algebra with maximal ideal $\mathfrak{m}$ and residue field isomorphic to $\mathbb{C}$. There is a block decomposition ([Fie03, Section 2.4])

$$\mathcal{O}_T = \Pi_\Lambda \mathcal{O}_{T,\Lambda}$$

indexed by certain sets Λ of weights (the highest weights of the simple objects in the block). The block $\mathcal{O}_{T,\Lambda}$ is the full subcategory of O_T given by all objects where the highest weights of any subquotient are contained in Λ. If all the weights of Λ are outside the critical hyperplanes then we call $\mathcal{O}_{T,\Lambda}$ *a block outside the critical hyperplanes*. If moreover T is a local deformation domain then Λ is the dot-orbit of any of its elements under the corresponding *integral* Weyl group $\mathcal{W}_T(\Lambda)$ (which is generated by reflections corresponding to real roots only, [Fie03, page 699]). If $T \to T'$ is a homomorphism of local deformation algebra domains, then the base change functor $T' \otimes_T \bullet$ maps the blocks of $\mathcal{O}_T$ to the blocks of $\mathcal{O}_{T'}$ ([Fie03, Corollary 2.10]). On the other hand, for $T' = T/\mathfrak{m}$ the resulting decomposition is exactly the block decomposition from [KK79, Theorem 2]. In case $T \to T'$ is a homomorphism of deformation algebras we denote by $\mathcal{O}_{T',\Lambda}$ the image of a block $\mathcal{O}_{T,\Lambda}$ under the base change functor. In general, this is not a block, but a direct sum of blocks ([Fie03, Lemma 2.9, Corollary 2.10]). For any $\mathcal{O}_{T,\Lambda}$ we denote by $\mathcal{M}_{T,\Lambda}$ the full subcategory of $\mathcal{O}_{T,\Lambda}$ given by all modules having a finite Verma flag.

3.3 Translation through walls

Let T be any local deformation algebra domain. Let $\mathcal{O}_{T,\Lambda}$, $\mathcal{O}_{T,\Lambda'}$ be two blocks, outside the critical hyperplanes. Let $\lambda \in \Lambda$, $\lambda' \in \Lambda'$. Assume that

(1) $\lambda - \lambda'$ is integral and there is a dominant weight ν in the $\mathcal{W}$-orbit of $\lambda - \lambda'$ (in particular $\mathcal{W}_T(\Lambda) = \mathcal{W}_T(\Lambda')$),

(2) λ and λ' lie in the closure of the same Weyl chamber,

(3) under the dot-action, the $\mathcal{W}_T(\Lambda)$-stabiliser of λ is contained in the $\mathcal{W}_T(\Lambda')$-stabiliser of λ',

then there are the translation functors *onto the wall* and *out of the wall*

$$\theta_{on} : \mathcal{M}_{T,\Lambda} \to \mathcal{M}_{T,\Lambda'} \qquad \theta_{out} : \mathcal{M}_{T,\Lambda'} \to \mathcal{M}_{T,\Lambda}$$

as defined in [Fie03]. The definition of θ_{on} is completely analogous to the finite dimensional case, and given by tensoring with the simple integrable highest weight module $L(\nu)$ and projection onto the block. The functor θ_{out} is defined by taking a limit of certain truncations of the functor tensoring with the restricted dual of $L(\nu)$ (see [Fie03, 4.1]). If λ' lies exactly on the s-wall, i.e. the $\mathcal{W}_T(\Lambda')$-stabiliser of λ' is $\{e, s\}$ and λ is regular, then we have the translation functor $\theta_s = \theta_{out}\theta_{on}$ through the s-wall. Note that translation functors commute with base change in the sense of [Fie03, 5.2]. In particular, it gives translation functors for any $S_{(0)}$-algebra T'.

3.4 The fake antidominant projective module

Let $\mathcal{O}_{T,\Lambda} \subset \mathcal{O}_T$ denote a regular block where all the corresponding weights are outside the critical hyperplanes. In general, $\mathcal{O}_{T,\Lambda}$ does not have enough projectives. In particular, in the Kac-Moody case, Soergel's famous "antidominant projective module" does not need to exist. However, there are ([Fie03, Theorem 2.7]) enough projectives in the truncated categories $\mathcal{O}^{\nu}_{T,\Lambda}$ given by all objects in $\mathcal{O}_{T,\Lambda}$ whose weights are $\leq \nu$ (with $\nu \in \mathfrak{h}^*$ fixed). If Λ contains an antidominant weight λ, we denote by $P^n_T(\lambda)$ the projective cover of the simple module $L_T(\lambda)$ in $\mathcal{O}^{\lambda+n\chi}_{T,\Lambda}$, where χ is the sum of all simple roots. We choose a compatible system of surjections $p_{m,n} : P^m_T(\lambda) \twoheadrightarrow P^n_T(\lambda)$ for $m \geq n$ and denote $P^{\infty}_T(\lambda) = \varprojlim P^n(\lambda)$. By definition, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathfrak{g}_T}(P^{\infty}_T(\lambda), \varprojlim P^n_T(\lambda)) \cong \varprojlim \mathrm{Hom}_{\mathfrak{g}}(P^{\infty}_T(\lambda) P^n_T(\lambda)).$$

Since the largest quotient of $P^{\infty}_T(\lambda)$ contained in $\mathcal{O}^{\lambda+n\chi}$ is $P^n_T(\lambda)$, the induced inclusion

$$\mathrm{Hom}_{\mathfrak{g}_T}(P^n_T(\lambda), P^n_T(\lambda)) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}_T}(P^{\infty}_T(\lambda), P^n_T(\lambda))$$

is in fact an isomorphism. Therefore, we have canonically

$$\begin{aligned} \varprojlim \mathrm{End}_{\mathfrak{g}_T}(P^n_T(\lambda)) &\cong \varprojlim \mathrm{Hom}_{\mathfrak{g}_T}(P^{\infty}_T(\lambda), P^n_T(\lambda)) \\ &\cong \mathrm{Hom}_{\mathfrak{g}_T}(P^{\infty}_T, \varprojlim P^n_T(\lambda)) \\ &= \mathrm{End}_{\mathfrak{g}_T}(P^{\infty}_T(\lambda)). \end{aligned} \tag{3.1}$$

If $F : \mathcal{M}_{T,\Lambda} \to \mathcal{M}_{T,\Lambda}$ is a composition of translations through walls we set $FP^{\infty}_T(\lambda) = \varprojlim FP^n_T(\lambda)$, where the defining maps are the $F(p_{m,n})$ for $m \geq n$.

3.5 Blocks containing an antidominant weight

The definition of translation functors through walls is based on the existence of some dominant weight ν as stated above. Such a dominant weight need not to exist in general for any block $\mathcal{O}_{T,\Lambda}$ outside the critical hyperplanes. However, in these cases there will be an antidominant weight $\lambda \in \Lambda$. We will study such blocks now. Let T be a local deformation algebra and let $\mathcal{O}_{T,\Lambda}$ be a regular block outside the critical hyperplanes. Let us assume Λ to have an antidominant weight λ. Let $\tau(\Lambda) = \{-2\rho - \lambda \mid \lambda \in \Lambda\}$. In particular, $\tau(\Lambda)$ contains a dominant weight, and therefore $\mathcal{O}_{\Lambda,T}$ has enough projectives. We consider the (Chevalley-)anti-automorphism σ interchanging root spaces $\mathfrak{g}_{\alpha}$ and $\mathfrak{g}_{-\alpha}$ and the principal anti-automorphism $\gamma : S \to S$, $h \mapsto -h$ for any $h \in \mathfrak{h}$. For a $\mathfrak{g}$-module M, we denote by ${}^{\sigma}M$ the space M but with σ-twisted $\mathfrak{g}$-action. Likewise, given an S-module N, the symbol ${}^{\gamma}N$ denotes N with γ-twisted action of S. The *semi-infinite bimodule* $S_{-2\rho}$ with respect to the semi-infinite character -2ρ is defined in [Soe97a] (and relies on the work of Arkhipov, Frenkel and Voronov). It provides an equivalence of categories

$$\tau : \mathcal{M}_{T,\Lambda} \tilde{\to} \mathcal{M}^{\mathrm{opp}}_{{}^{\gamma}T,\tau(\Lambda)} \tag{3.2}$$

which maps short exact sequences to such ([Soe97a]). More precisely, the equivalence is constructed as follows: We consider the functor

$$\mathcal{T}_{-2\rho} : \quad M \mapsto \quad {}^{\sigma}\big((S_{-2\rho} \otimes_{\mathcal{U}} M)^{\circledast}\big)$$

from the category of $\mathfrak{g}_T$-modules whose weight spaces are free T-modules of finite rank. Hence $(S_{-2\rho} \otimes_{\mathcal{U}} M)^{\circledast} := \bigoplus_{\lambda\in\mathfrak{h}^*} \mathrm{Hom}_T((S_{-2\rho} \otimes_{\mathcal{U}} M)_\lambda, T)$ is again a free T-module. The natural right $\mathfrak{g}$-action becomes a left $\mathfrak{g}$-action after twisting with σ. Hence $\mathcal{T}_{-2\rho}(M)$ is a $\mathfrak{g}_T$-module. The arguments in [Soe97a] show that $\mathcal{T}_{-2\rho}$ restricts to an equivalence (3.2), called the *tilting equivalence*, sending $\Delta_T(\mu)$ to $\Delta_{{}^{\gamma}T}(\tau(\mu))$ for any $\mu \in \Lambda$. Note that if $T = S_{(0)}$ or $T = \mathbb{C}$, then ${}^{\gamma}T \cong T$ as S-modules. Therefore, if $P \in \mathcal{O}_{T,\tau(\Lambda)}$ is projective and finitely generated, hence in $\mathcal{M}_{T,\Lambda}$, then $\tau(P) \in \mathcal{M}^{\mathrm{opp}}_{T,\Lambda}$ is *tilting*, i.e. it has a finite Verma flag and

$$\mathrm{Ext}^1_{\mathcal{O}_{T,\Lambda}}(\Delta(\mu), \tau(P)) = \mathrm{Ext}^1_{\mathcal{O}_{{}^{\gamma}T,\tau(\Lambda)}}(P, \Delta(\tau(\mu)) = 0$$

for any $\mu \in \Lambda$. It follows directly from the definitions that $\mathcal{T}_{-2\rho}$ commutes with base change. Via these tilting equivalences the translation functors θ_s from Section 3.2 give rise to translation functors θ_s for blocks containing an antidominant weight.

In the following an *antidominant block* means a regular block $\mathcal{O}_{T,\Lambda}$ outside the critical hyperplanes such that Λ contains an antidominant weight which we denote by λ.

We first need an analogue of Proposition 1.2, namely that the simple objects occurring in the socle of a Verma module in $\mathcal{O}^{\mathfrak{p}}$ are of maximal Gelfand-Kirillov.

Lemma 3.1. *Let $\mathcal{O}_{\mathbb{C},\Lambda}$ be an antidominant block. Let $X \in \mathcal{M}_{\mathbb{C},\Lambda}$. Then $\mathrm{Hom}_{\mathfrak{g}}\big(P^{\infty}_{\mathbb{C}}(\lambda), X'\big) \neq 0$ for any submodule X' of X.*

Proof. As a submodule of a module with Verma flag, X' contains a Verma module, hence also $\Delta_{\mathbb{C}}(\lambda)$ by [Fie04, Theorem 3.10]. □

The following result is again well-known in the finite dimensional situation (see e.g. [Jan83, 4.13 (1)]):

Lemma 3.2. *Let T be a local deformation algebra domain or $T = \mathbb{C}$. Let $\mathcal{O}_{T,\Lambda}$ be an antidominant block. Then $P^n_T(\lambda)$ has a finite Verma flag and for any $\mu \in \Lambda$ we have $(P^n_T(\lambda) : \Delta_T(\mu)) = 1$ if $n \gg 0$.*

Proof. The existence of a finite Verma flag is given by [Fie03, Theorem 2.7]. The multiplicity formula is [Fie04, Lemma 3.8]. □

3.6 The centre

Recall from Section 1 that the centre C of the algebra $A^{\mathfrak{b}}$, or equivalently the centre of the category $\mathcal{O}_0^{\mathfrak{b}}$, is isomorphic to the endomorphism ring of the full tilting module T, which is Soergel's "antidominant projective module", and the classification theorem of projective functors can be obtained by considering certain special C-bimodules. We would like to generalise this approach to the Kac-Moody case. Therefore we first describe the centre of the deformed categories $\mathcal{O}_{T,\Lambda}$ (generalising the main result of [Fie03]).

Lemma 3.3. *Let $T = S_{(0)}$ or $T_{\mathbb{C}}$. Let $\mathcal{O}_{T,\Lambda}$ be an antidominant block and $s \in \mathcal{W}(\Lambda)$ a simple reflection. Then $\theta_s P_T^\infty(\lambda) \cong P_T^\infty(\lambda) \oplus P_T^\infty(\lambda)$.*

Proof. By definition of the translation functors, $\theta_s P_T^n(\lambda)$ is projective in a suitable truncation $\mathcal{O}'_{T,\Lambda}$ of $\mathcal{O}_{T,\Lambda}$. Since $T = S_{(0)}$ or $T = \mathbb{C}$ the functor θ_s is self-adjoint ([Fie03, Corollary 5.10]). Since $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}_T}(P_{\mathbb{C}}^n(\lambda), \theta_s \Delta_{\mathbb{C}}(\lambda)) = 2$ by [Fie03, Corollary 5.10, Proposition 4.1] and [Fie04, Remark 3.9 (2)] we get that the indecomposable cover of $\Delta_{\mathbb{C}}(\lambda) \in \mathcal{O}'_{\mathbb{C},\Lambda}$ occurs with multiplicity 2 as a direct summand of $\theta_s P_{\mathbb{C}}^n(\lambda)$. Therefore, in $\theta_s P_{S_{(0)}}^n(\lambda)$, the indecomposable cover of $\Delta_{S_{(0)}}(\lambda)$ occurs with multiplicity 2 as well ([Fie03, Proposition 2.6, Lemma 5.4]), since the blocks in $\mathcal{O}_{\mathbb{C}}$ are the specialisations of the blocks of $\mathcal{O}_{S_{(0)}}$. If another direct summand occurs, than it has to occur for any $m > n$. However, Lemma 3.2 says that any Verma module occurs in $\theta_s P_T^\infty(\lambda)$ once, and [Fie03, Proposition 4.1] implies that any Verma module occurs in $\theta_s P_T^\infty(\lambda)$ with multiplicity 2. It follows $\theta_s P_T^\infty(\lambda) \cong P_T^\infty(\lambda) \oplus P_T^\infty(\lambda)$. □

Remark 3.4. If T is any local deformation algebra then $\theta_s P_T^\infty(\lambda)$ is isomorphic to a finite direct sum of fake antidominant projective modules $P_T^\infty(\lambda')$, each occurring with finite multiplicity. ([Fie03, Lemma 5.4], [Fie04, Remark 3.9]).

Theorem 3.5. *Let $\mathcal{O}_{S_{(0)},\Lambda}$ be an antidominant block. Let $T = \mathbb{C}$ or $S_{(0)} \to T$ be a morphism of local deformation algebras. There is a natural isomorphism*

$$\mathcal{I} = \mathcal{I}_{T,\Lambda} : \quad \operatorname{End}(\mathrm{id}_{\mathcal{O}_{T,\Lambda}}) \quad \cong \quad \operatorname{End}_{\mathfrak{g}_T}(T \otimes_{S(0)} P_{S(0)}^\infty(\lambda))$$

Proof. Using the identifications (3.1) we claim that $\phi \mapsto \phi_\infty := \{\phi_{P_T^n(\lambda)}\}$ defines the required isomorphism.

Injectivity: If $T = \mathbb{C}$ then the injectivity follows directly from Lemma 3.1. For the general case we have to work more. Assume $\phi_\infty = 0$. Assume now $T = S_{(0)}$ and let $X \in \mathcal{O}_{T,\Lambda}$ be a tilting module (having a finite Verma flag). Let us assume X is indecomposable. By [Fie04, Section 4] X is a direct summand of some $F\Delta_T(\lambda)$, where F is a composition of translations through walls. Since $P_T^n(\lambda)$ has a finite Verma flag with $\Delta_T(\lambda)$ at the top, and F preserves short exact sequences of modules with finite Verma flag ([Fie03, Proposition 4.1]), $FP_T^n(\lambda)$ surjects onto X. By Lemma 3.3, a (finite) direct sum of $P_T^\infty(\lambda)$ surjects onto X. Hence $\phi_X = 0$. Since $\mathcal{O}_{T,\Lambda}$ is generated by modules having a finite Verma flag ([Fie03, Theorem 2.7]), $\mathcal{I}$ is injective. The general case follows by base change using Remark 3.4.

Surjectivity: Via restriction the modules $\{\mathcal{Z}^n_{T,\Lambda} := \operatorname{End}(\mathrm{id}_{\mathcal{O}^{\lambda+n\chi}_{T,\Lambda}})\}_{n\in\mathbb{N}}$ form a projective system and $\mathcal{Z}_{T,\Lambda} := \operatorname{End}(\mathrm{id}_{\mathcal{O}_{T,\Lambda}}) \cong \varprojlim \mathcal{Z}^n_{T,\Lambda}$. It is sufficient to show that $\operatorname{End}(\mathrm{id}_{\mathcal{O}^{\lambda+n\chi}_{T,\Lambda}}) \hookrightarrow \operatorname{End}_{\mathfrak{g}_T}(T\otimes_{S(0)} P^n_{S(0)}(\lambda))$, $\phi\mapsto\phi_{P^n_T(\lambda)}$ defines an isomorphism for any $n\in\mathbb{N}$. If $T=S_{(0)}$ then the statement is true by [Fie04, Lemma 3.12]. Let now $Q=\operatorname{Quot}(S_{(0)})$ be the quotient field of $S_{(0)}$. Since there are no extensions between Verma modules, there are canonical isomorphisms

$$\operatorname{End}\left(\mathrm{id}_{\mathcal{O}^{\lambda+n\chi}_{Q,\Lambda}}\right) \cong \prod_{\lambda+n\chi\geq\mu\in\Lambda} \operatorname{End}_{\mathfrak{g}_Q}\left(\Delta_Q(\mu)\right) \cong \prod_{\lambda+n\chi\geq\mu\in\Lambda} Q.$$

On the other hand

$$\begin{aligned}\operatorname{End}_{\mathfrak{g}_Q}\left(Q\otimes_{S(0)} P^n_{S(0)}(\lambda)\right) &\cong \operatorname{End}_{\mathfrak{g}_Q}\Big(\prod_{\lambda+n\chi\geq\mu\in\Lambda}\Delta_Q(\mu)\Big)\\ &\cong \prod_{\lambda+n\chi\geq\mu\in\Lambda}\operatorname{End}_{\mathfrak{g}_Q}\left(\Delta_Q(\mu)\right)\end{aligned}$$

by Lemma 3.2. Since the maps in question are morphisms of free Q-modules of finite rank, $\mathcal{I}$ becomes an isomorphism $\mathcal{I}_{Q,\Lambda}$ at the generic point. We have to show that it also becomes an isomorphism $\mathcal{I}_{\mathbb{C},\Lambda}$ at the closed point to get the surjectivity in general. For this let $f\in\operatorname{End}_{\mathfrak{g}_\mathbb{C}}(\mathbb{C}\otimes_{S(0)} P^\infty_{S(0)})$. Recall that specialisation defines an isomorphism $\operatorname{End}_{\mathfrak{g}_\mathbb{C}}(\mathbb{C}\otimes_{S(0)} P^\infty_{S(0)}(\lambda)) \cong \mathbb{C}\otimes_{S(0)} \operatorname{End}_{\mathfrak{g}_{S_{(0)}}}(P^\infty_{S(0)}(\lambda))$ ([Fie03, Proposition 2.4] or [Fie04, Remark 3.9]) under which f corresponds to some $1\otimes g$. Since $\mathcal{I}_{S_{(0)},\Lambda}$ is an isomorphism, there exists some $\phi\in\operatorname{End}(\mathrm{id}_{\mathcal{O}_{S_{(0)},\Lambda}})$ such that $\mathcal{I}_{S_{(0)},\Lambda}(\phi)=g$. Then $1\otimes\phi\in\operatorname{End}(\mathrm{id}_{\mathcal{O}_{\mathbb{C},\Lambda}})$ ([Fie03, Proposition 3.1]) and by definition of $\mathcal{I}$ we get $\mathcal{I}_{\mathbb{C},\Lambda}(1\otimes\phi)=\mathcal{I}_{S_{(0)},\Lambda}(1\otimes\phi)=1\otimes g=f$. This means $\mathcal{I}_{\mathbb{C},\Lambda}$ is surjective. The statement of the theorem follows. □

We denote $\mathcal{Z}_{T,\Lambda}=\operatorname{End}(\mathrm{id}_{\mathcal{O}_{T,\Lambda}})$, the centre of $\mathcal{O}_{T,\Lambda}$.

Corollary 3.6. *Let $\mathcal{O}_{S_{(0)},\Lambda}$ be an antidominant block. Let $T=\mathbb{C}$ or $S_{(0)}\to T$ be a morphism of local deformation algebras. Then there is a natural isomorphism*

$$\mathcal{Z}_{T,\Lambda}\cong T\otimes_{S(0)}\mathcal{Z}_{S_{(0)},\Lambda}.$$

Proof. Let $n\in\mathbb{N}$. There are isomorphisms of rings

$$T\otimes_{S(0)}\operatorname{End}_{\mathfrak{g}_{S(0)}}\left(P^n_{S(0)}(\lambda)\right)\cong\operatorname{End}_{\mathfrak{g}_T}\left(T\otimes_{S(0)}P^n_{S(0)}(\lambda)\right)$$

by [Fie03, Proposition 2.4] and [Fie03, Proposition 2.6]. Taking limits the theorem gives the statement. □

3.7 The structure theorem

Let $\mathcal{O}_{S_{(0)},\Lambda}$ be an antidominant block. Let $S_{(0)}\to T$ be a morphism of local deformation algebras. We consider the functor $\mathbb{V}_T=\operatorname{Hom}_{\mathfrak{g}_T}(T\otimes_{S(0)}P^\infty_{S(0)}(\lambda),\bullet):$

$\mathcal{O}_{T,\Lambda} \to \text{mod-}\mathcal{Z}_{T,\Lambda}$. Note that with the assumptions of Corollary 3.6 and the formulas (3.1) there are canonical isomorphisms of $\mathcal{Z}_{T,\Lambda}$-modules

$$\begin{aligned} \mathbb{V}_{S(0)} \varprojlim P^n_{S(0)}(\lambda) &\cong \varprojlim \mathbb{V}_{S(0)} P^n_{S(0)} &&\cong \varprojlim \operatorname{End}_{\mathfrak{g}_{S(0)}}(P^n_{S(0)}(\lambda)) \\ &\cong \operatorname{End}_{\mathfrak{g}_{S(0)}}(P^\infty_{S(0)}) &&\cong \mathcal{Z}_{S(0),\Lambda}. \end{aligned} \tag{3.3}$$

Using Corollary 3.6, specialisation gives an isomorphism

$$\begin{aligned} \mathbb{V}_T \varprojlim (T \otimes_{S(0)} P^n_{S(0)}(\lambda)) &\cong \varprojlim \mathbb{V}_{S(0)}(T \otimes_{S(0)} P^n_{S(0)}) \\ &\cong T \otimes_{S(0)} (\varprojlim \operatorname{End}_{\mathfrak{g}_{S(0)}}(P^n_{S(0)}(\lambda))) \\ &\cong T \otimes_{S(0)} \operatorname{End}_{\mathfrak{g}_{S(0)}}(P^\infty_{S(0)}) \\ &\cong T \otimes_{S(0)} \mathcal{Z}_{S(0),\Lambda} \\ &\cong \mathcal{Z}_{T,\Lambda}. \end{aligned} \tag{3.4}$$

Recall that the deformed Verma modules in an antidominant block $\mathcal{O}_{S_{(0)},\Lambda}$ are exactly the ones with highest weight in $\Lambda = \mathcal{W}(\Lambda) \cdot \lambda$. In [Fie03, Theorem 3.6] one can find an explicit description of the centre $\mathcal{Z}_{S_{(0)},\Lambda}$ as a subring of $\Pi_{w \in \mathcal{W}(\Lambda)} S_{(0)}$ by looking at the acting on each deformed Verma module. Together with Corollary 3.6 we get a concrete description of $\mathcal{Z}_{\mathbb{C},\Lambda}$. From this description we also get a natural right action of $\mathcal{W}(\Lambda)$ on $\mathcal{Z} := \mathcal{Z}_{S_{(0)},\Lambda}$. For any simple reflection $s \in \mathcal{W}(\Lambda)$ we denote by $\mathcal{Z}^s$ its invariants and get the following important result

Lemma 3.7. *$\mathcal{Z}$ is a free $\mathcal{Z}^s$-module and gives rise to a self-adjoint functor functor*

$$\bullet_{\mathcal{Z}^s} \mathcal{Z} : \text{mof-}\mathcal{Z} \to \text{mof-}\mathcal{Z}.$$

Moreover, there is an isomorphism of functors $\mathbb{V}_{S(0)} \theta_s(\bullet) \cong \mathbb{V}_{S(0)}(\bullet) \otimes_{\mathcal{Z}^s} \mathcal{Z}$.

Proof. Set $R = S_{(0)}$. The following argument is due to P. Fiebig: Let α be the simple root corresponding to s. For $w \in \mathcal{W}$ such that $w \cdot \lambda \in \Lambda$ we set $z_w = h_{w(\alpha)}$. In particular, $z_w = -z_{ws}$ and also $\{z_w\} \in \mathcal{Z}$ by [Fie03, Theorem 3.6]. Let now $a = \{a_w\}$ be an element of $\mathcal{Z}$. We claim that there are uniquely defined elements $c_+, c_- \in \mathcal{Z}^s$ such that $a = c_+ + c_- z$ and (hence 1, z the desired basis). From the general equality $ws = s_{w\alpha} w$, we get $a_{ws} = a_{s_{w(\alpha)}} \equiv a_w \mod z_w$ by [Fie03, Theorem 3.6]. Let now r_1, $r_2 \in R$ such that $r_1 \equiv r_2 \mod h$ for some $0 \neq h \in \mathfrak{h}$. Then the equation $(r_1, r_2) = c_+(1,1) + c_-(h, -h)$ has the unique solution $c_\mp = \frac{1}{2h}(r_1 \mp r_2)$. By [Fie03, Theorem 3.6] the elements x_+, x_- exist and are contained in $\mathcal{Z}$, because $a \in \mathcal{Z}^s$. The self-adjointness follows then as in [Soe05, Proposition 5.10]. The last statement is [Fie04, Theorem 4.1]. □

Lemma 3.8. *Let $\mathcal{O}_{S_{(0)},\Lambda}$ be an antidominant block. Let $T = \mathbb{C}$ or let $S_{(0)} \to T$ be a morphism of local deformation algebras. Then $\mathbb{V}_T$ induces an isomorphism*

$$\Phi : \operatorname{Hom}_{\mathfrak{g}_T}(X, Y) \cong \operatorname{Hom}_{\mathcal{Z}_{T,\Lambda}}(\mathbb{V}_T X, \mathbb{V}_T Y)$$

for $X = T \otimes_{S(0)} P^\infty_{S(0)}(\lambda)$ and any $Y \in \mathcal{O}_{T,\Lambda}$ or $Y = T \otimes_{S(0)} F P^\infty_{S(0)}(\lambda)$, where F is a finite composition of translation functors through walls.

Proof. Let first $Y \in \mathcal{O}_{T,\Lambda}$. By definition of $\mathbb{V}_T$ we have

$$\mathbb{V}_T Y = \mathrm{Hom}_{\mathfrak{g}_T}(T \otimes_{S(0)} P^{\infty}_{S(0)}(\lambda), Y) \to \mathrm{Hom}_{\mathcal{Z}_{T,\Lambda}}(\mathbb{V}_T X, \mathbb{V}_T Y) \cong \mathbb{V}_T Y,$$

the latter by evaluating f at the identity morphism id. This composition is mapping $f \in \mathbb{V}_T Y$ to $\mathbb{V}_T(f)(\mathrm{id}) = f \circ \mathrm{id} = f$, hence the middle arrow is an isomorphism and the statement follows for any $Y \in \mathcal{O}_{T,\Lambda}$. Then the lemma follows by taking limits. $\square$

The following is a crucial refinement of [Fie04, Theorem 3.25].

Theorem 3.9 (Structure theorem). *Let $\mathcal{O}_{S_{(0)},\Lambda}$ be an antidominant block. Let $T = \mathbb{C}$ or let $S_{(0)} \to T$ be a morphism of local deformation algebras. Let X, $Y \in \mathcal{O}_{T,\Lambda}$ be tilting modules. Then $\mathbb{V}_T$ induces a natural inclusion*

$$\Phi : \mathrm{Hom}_{\mathfrak{g}_T}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{Z}_{T,\Lambda}}(\mathbb{V}_T X, \mathbb{V}_T Y) \tag{3.5}$$

which is an isomorphism (at least) if $T = S(0)$ or $T = \mathbb{C}$.

Proof. Recall from [Fie04] that any indecomposable tilting module $X \in \mathcal{O}_{T,\Lambda}$ is a direct summand of some $T \otimes_{S(0)} (F\Delta_{S(0)}(\lambda))$ for some composition F of translations through walls. Hence it is enough to show the statement for $X = T \otimes_{S(0)} (F\Delta_{S(0)}(\lambda))$. Choose compatible surjections $P^n_{S(0)}(\lambda) \to \Delta_{S(0)}(\lambda)$ for $n \geq 0$. Then $FP^n_{S(0)}(\lambda) \to F\Delta_{S(0)}(\lambda)$ is again surjective ([Fie03, Proposition 4.1]) and so is the specialisation $T \otimes_{S(0)} (FP^n_{S(0)}(\lambda)) \to T \otimes_{S(0)} (F\Delta_{S(0)}(\lambda))$, and it stays surjective if we apply the exact functor $\mathbb{V}_T$. Thanks to Lemma 3.3 it is enough to verify (3.5) for $X = T \otimes_{S(0)} P^{\infty}_{S(0)}(\lambda)$. This is however Lemma 3.8. Let now $T = S_{(0)}$ or $T = \mathbb{C}$. Because of the self-adjointness of translations through walls [Fie04, Corollary 5.10, Proposition 3.11 (4)] and Lemma 3.7 we may assume that F is isomorphic to the identity, hence $X \cong \Delta_T(\lambda)$. Let I be the maximal ideal of $\mathcal{Z}_{T,\Lambda} = \mathrm{End}_{\mathfrak{g}_T}(P^{\infty}_T(\lambda))$. Then $\mathrm{Hom}_{\mathfrak{g}_T}(X, Y)$ can be identified with the space $\{f \in \mathrm{Hom}_{\mathfrak{g}_T}(P^{\infty}_T(\lambda), Y) \mid f \circ g = 0, g \in I\}$ and $\mathrm{Hom}_{\mathfrak{g}_T}(\mathbb{V}_T X, \mathbb{V}_T Y)$ can be identified with the space $\{f \in \mathrm{Hom}_{\mathcal{Z}_{T,\Lambda}}(\mathbb{V}_T P^{\infty}_T(\lambda), Y) \mid f \circ g = 0, g \in I\}$. So, the theorem follows from Lemma 3.8. $\square$

3.8 The combinatorics of natural transformations

Our next step is to prove a generalisation of Theorem 1.8 for Kac-Moody algebras. We start with some preparations. From now on we fix an antidominant block $\mathcal{O}_{S_{(0)},\Lambda}$. Let $T = S(0)$ or $T = \mathbb{C}$ and denote $P^m = P^m_T(\lambda)$ and $P = P^{\infty}_T(\lambda)$. Note that, for any finite composition F of translation functors through walls, the module FP has naturally a $\mathcal{Z}_{T,\Lambda}$-module structure, by definition of the centre. This action commutes with the second left $\mathcal{Z}_{T,\Lambda}$-structure given by $z.m = F(z)(m)$ for any $z \in \mathcal{Z}_{T,\Lambda} = \mathrm{End}_{\mathfrak{g}_T}(P)$, $m \in FP$. Via the functor $\mathbb{V}_T$ the first left action converts into the usual right $\mathcal{Z}_{T,\Lambda}$-action on $\mathbb{V}_T FP$ and the second actions turns into the left $\mathcal{Z}_{T,\Lambda}$-action $z.f = F(z_P) \circ f$. We have a projective system

$$\left\{H_m := \mathrm{Hom}_{\mathfrak{g}_T}(P, FP^m(\lambda)) = \mathbb{V}_T FP^m(\lambda)\right\}_{n \in \mathbb{N}}$$

given by the maps $F(p_{m,n}) \circ \cdot : H_m \to H_n$ for $m \geq n$. Set $\mathbb{V}_T FP(\lambda) = \mathrm{Hom}_{\mathfrak{g}_T}\left(P, \varprojlim FP^n(\lambda)\right) \cong \varprojlim \mathrm{Hom}_{\mathfrak{g}_T}(P(\lambda), FP^n(\lambda))$. We get the following generalisation of Theorem 1.8:

Theorem 3.10. *Let $T = S_{(0)}$ or $T = \mathbb{C}$ and let $\mathcal{O}_{T,\Lambda}$ be an antidominant block. Let F, $G : \mathcal{M}_{T,\Lambda} \to \mathcal{M}_{T,\Lambda}$ be finite compositions of translations through walls. There is a natural isomorphism of vector spaces (even of rings if $F = G$)*

$$\mathrm{Hom}(F, G) \cong \mathrm{Hom}_{\mathcal{Z}_{T,\Lambda}\text{-}\mathrm{mod}\text{-}\mathcal{Z}_{T,\Lambda}}(\mathbb{V}_T FP^\infty(\lambda), \mathbb{V}_T GP^\infty(\lambda))$$

Proof. We claim that $\phi \mapsto a^\phi \in \mathrm{H}$ where $(a^\phi)_n(\{g_j\}_{j\in\mathbb{N}}) = \phi_n \circ g_n$ with $\phi_n = \phi_{P^n}$ defines the required isomorphism, where H is defined as follows

$$\begin{aligned} & \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathbb{V}_T FP, \mathbb{V}_T GP) \\ = \; & \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}\left(\mathrm{Hom}_{\mathfrak{g}_T}\left(P, \varprojlim_m FP^m\right), \mathrm{Hom}_{\mathfrak{g}_T}\left(P, \varprojlim_n GP^n\right)\right) \\ = \; & \varprojlim_n \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}\left(\varprojlim_m \mathrm{Hom}_{\mathfrak{g}_T}\left(P, FP\right), \mathrm{Hom}_{\mathfrak{g}_T}\left(P, GP\right)\right) =: H \end{aligned}$$

Well-defined: Since ϕ is a natural transformation, one easily deduces that a^ϕ is a $\mathcal{Z}_{T,\Lambda}$-bimodule morphism. For $m \geq n$ we have $G(p_{m,n}) \circ (a^\phi)_m(\{g_j\}_{j\in\mathbb{N}}) = G(p_{m,n}) \circ \phi_m \circ g_m = \phi_n \circ F(p_{m,n}) \circ g_m = \phi_n \circ g_n = (a^\phi)_n(\{g_j\}_{j\in\mathbb{N}})$ and our map is therefore well-defined.

Injectivity: Assume $a^\phi = 0$, in particular $\phi_n = 0$ for any $n \in \mathbb{N}$. For any tilting module $X \in \mathcal{O}_{T,\Lambda}$ we choose compatible surjections $p_n : P^n_{S(0)}(\lambda) \to \Delta_{S(0)}(\lambda)$ for $n \geq 0$. Then $F(p_n)$ is again surjective ([Fie03, Proposition 4.1]). Since ϕ is a natural transformation we get $\phi_X \circ F(p_n) = G(p_n) \circ \phi_n = 0$ for any $n > 0$, so $\phi_X = 0$. Since $\mathcal{M}_{T,\Lambda}$ is generated by tilting modules (see Section 3.5) the injectivity follows.

Surjectivity: Since (under the assumptions on T) any translation through the wall is self-adjoint ([Fie03, Corollary 5.10]), using Lemma 3.7 we are allowed to assume $F = \mathrm{id}$. Let $f \in \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathbb{V}_T P, \mathbb{V}_T GP)$. Let $X \in \mathcal{M}_{T,\Lambda}$. We choose a complex

$$\mathcal{K} : \oplus_{i\in I} P^\infty \to \oplus_{j\in J} P^\infty \to X \to 0$$

with finite sets I and J such that $\mathbb{V}\mathcal{K}$ is exact; in other words the homology of $\mathcal{K}$ does not contain $L_T(\lambda)$ as a composition factor. We get a commutative diagram of the form

$$\begin{tikzcd} \oplus_{i\in I}\mathbb{V}P \arrow[r] \arrow[d, "\oplus f"] & \oplus_{j\in J}\mathbb{V}P \arrow[r] \arrow[d, "\oplus f"] & \mathbb{V}X \arrow[r] \arrow[d, dotted, "f_X"] & 0 \\ \oplus_{i\in I}\mathbb{V}GP \arrow[r] & \oplus_{j\in J}\mathbb{V}GP \arrow[r] & \mathbb{V}GX \arrow[r] & 0 \end{tikzcd}$$

where the rows are complexes and where, by assumption, the first row is exact. Since f is a $\mathcal{Z}_{T,\Lambda}$-bimodule map, the left part of the diagram commutes inducing a unique morphism f_X as indicated. If X is tilting, then the Structure Theorem 3.9 induces a unique map $f_X \in \mathrm{Hom}_{\mathfrak{g}_T}(X, GX)$ which is natural. By standard arguments, this induces a natural transformation $\mathrm{id} \to G$ when restricted to the additive category of tilting modules. Since $\mathcal{M}_{T,\Lambda}$ is generated by tilting modules the surjectivity follows. □

3.9 Towards the classification theorem

We still fix an antidominant block $\mathcal{O}_{S_{(0)},\Lambda}$ and denote $R = S(0)$, $\mathcal{Z} = \mathcal{Z}_{R,\Lambda}$, and will use the notation from Section 1.7.

Proposition 3.11. *Let $[x] = s_1 s_2 \cdots s_r$ and $[y] = s_{r+1} s_{r+2} \cdots s_{r+q}$ be fixed compositions of simple reflections in $\mathcal{W}(\Lambda)$. Then the following hold*

a.) $\mathrm{Hom}_{\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(R), \mathcal{Z}_{[y]}(R)) = \mathrm{Hom}_{R\text{-}\mathrm{mod}\text{-}\mathcal{Z}}\, \mathrm{Hom}_Z(Z_{[x]}(R), Z_{[y]}(R))$*, where R acts in the latter by multiplication from the left hand side.*

b.) The space $\mathrm{Hom}_{R\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(Z_{[x]}(R), Z_{[y]}(R))$ *is a (graded) free left R-module of finite rank.*

c.) The canonical map defines an isomorphism of vector spaces

$$\mathbb{C} \otimes_R \mathrm{Hom}_{R\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(R), \mathcal{Z}_{[y]}(R)) \cong \mathrm{Hom}_{\mathbb{C}\otimes_R R\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(R \otimes_R \mathbb{C}), \mathcal{Z}_{[y]}(R \otimes_R \mathbb{C})).$$

d.) The canonical map defines an isomorphism

$$\begin{aligned} & \mathcal{Z} \otimes_R \mathrm{Hom}_{R\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(R), \mathcal{Z}_{[y]}(R)) \\ \cong\ & \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(\mathcal{Z} \otimes_R R), \mathcal{Z}_{[y]}(\mathcal{Z} \otimes_R R)) \\ =\ & \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(\mathcal{Z}), \mathcal{Z}_{[y]}(\mathcal{Z})). \end{aligned}$$

Proof. a.) The ring R is obviously included in the centre $Z = \mathcal{Z}_{\mathcal{O}_{R,\Lambda}}$ via the diagonal embedding $R \hookrightarrow \prod_{\mu \in \mathcal{W}(\Lambda)} R$, $r \mapsto \{r\}_\mu$. The image is s-invariant for any simple reflection s. The claim follows then from the definitions.

b.) We consider the following compositions of translations through walls $F_R = \theta_{s_r} \cdots \theta_{s_2} \theta_{s_1}$, $G_R = \theta_{s_{r+q}} \cdots \theta_{s_{r+2}} \theta_{s_{r+1}} : \mathcal{O}_{R,\Lambda} \to \mathcal{O}_{R,\Lambda}$ with its specialisations $F_{\mathbb{C}}$ and $G_{\mathbb{C}}$. The Structure Theorem 3.9 together with [Fie04, Theorem 4.1] provides a natural isomorphism of vector spaces, compatible with the left R-action, as follows

$$\mathrm{Hom}_{\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(R), \mathcal{Z}_{[y]}(R)) \cong \mathrm{Hom}_{\mathfrak{g}_R}(F_R \Delta_R(\lambda), G_R \Delta_R(\lambda)).$$

The latter is a free R-module of finite rank, because via the tilting equivalence from Section 3.5 we are in the situation of [Fie03, Proposition 2.4] where it is shown that the morphism spaces between truncated projective modules are free R-modules of finite rank. Using part (a) we are done.

c.) We claim that there are natural isomorphisms

$$\begin{aligned}
& \mathbb{C} \otimes_R \mathrm{Hom}_{\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(R), \mathcal{Z}_{[y]}(R)) \\
\overset{(1)}{\cong}\ & \mathbb{C} \otimes_R \mathrm{Hom}_{\mathfrak{g}_R}\big(F_R\Delta_R(\lambda), G_R\Delta_R(\lambda)\big) \\
\overset{(2)}{\cong}\ & \mathrm{Hom}_{\mathfrak{g}_\mathbb{C}}\big(F_\mathbb{C}\Delta_\mathbb{C}(\lambda), G_\mathbb{C}\Delta_\mathbb{C}(\lambda)) \\
\overset{(3)}{\cong}\ & \mathrm{Hom}_{\mathrm{mod}\text{-}\mathbb{C}\otimes_R\mathcal{Z}}\big(\mathbb{V}_\mathbb{C}(F_\mathbb{C}\Delta_\mathbb{C}(\lambda)), \mathbb{V}_\mathbb{C}(G_\mathbb{C}\Delta_\mathbb{C}(\lambda))) \\
\overset{(4)}{\cong}\ & \mathrm{Hom}_{\mathrm{mod}\text{-}\mathbb{C}\otimes_R\mathcal{Z}}\big(\mathbb{V}_R(F_R\Delta_R(\lambda)) \otimes_R \mathbb{C}, \mathbb{V}_R(G_R\Delta_R(\lambda)) \otimes_R \mathbb{C}) \\
\cong\ & \mathrm{Hom}_{\mathbb{C}\otimes_R R\text{-}\mathrm{mod}\text{-}\mathcal{Z}}(\mathcal{Z}_{[x]}(R \otimes_R \mathbb{C}), \mathcal{Z}_{[y]}(R \otimes_R \mathbb{C})).
\end{aligned}$$

The isomorphism (1) follows from the Structure Theorem 3.9, Lemma 3.7 and Part (b). The isomorphism (2) exists by [Fie03, Proposition 2.4, Corollary 5.11] and the tilting equivalence. Thanks again to the Structure Theorem and Corollary 3.6 we have (3). Finally (4) is given by [Fie04, Proposition 3.11]. Using part (a), the proposition follows.

d.) This is completely analogous to [Soe92, Lemma 12].

□

3.10 The classification of projective functors

In this section we formulate and prove the classification theorems for projective functors. The crucial result is given by the following description of morphisms between compositions of translation functors as morphisms between translated deformed Verma modules, where the deformation ring is the centre of the original category.

Theorem 3.12. *Let $T = S_{(0)}$ or $T = \mathbb{C}$ and let $\mathcal{O}_{T,\Lambda}$ be an antidominant block with centre $\mathcal{Z}_T$. Let $F_T, G_T : \mathcal{O}_{T,\Lambda} \to \mathcal{O}_{T,\Lambda}$ be compositions of translation through walls with corresponding compositions $F_{\mathcal{Z}_T}, G_{\mathcal{Z}_T} : \mathcal{O}_{\mathcal{Z}_T,\Lambda} \to \mathcal{O}_{\mathcal{Z}_T,\Lambda}$. Then there is an isomorphism of vector spaces (or even of rings if $F = G$)*

$$\mathrm{Hom}(F_T, G_T) \cong \mathrm{Hom}_{\mathfrak{g}_{\mathcal{Z}_T}}(F_{\mathcal{Z}_T}\Delta_{\mathcal{Z}_T}(\lambda), G_{\mathcal{Z}_T}\Delta_{\mathcal{Z}_T}(\lambda)).$$

Proof. Let first be $T = S_{(0)} = R$. Let $\hat{F}$, $\hat{G}$ denote the functors obtained from Lemma 3.7 such that $\mathbb{V}_R F \cong \hat{F}\mathbb{V}_R$ and $\mathbb{V}_R G \cong \hat{G}\mathbb{V}_R$. Then we have isomorphisms of vector spaces (or rings)

$$
\begin{array}{lll}
 & \mathrm{Hom}(F_R, G_R) & \\
\cong & \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}\big(\mathbb{V}_R F_R P_R^\infty(\lambda), \mathbb{V}_R G_R P_R^\infty(\lambda)\big) & \text{(Theorem 3.10)} \\
\cong & \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}\big(\hat{F}_R \mathbb{V}_R P_R^\infty(\lambda), \hat{G}_R \mathbb{V}_R P_R^\infty(\lambda)\big) & \\
\cong & \mathrm{Hom}_{\mathcal{Z}\text{-}\mathrm{mod}\text{-}\mathcal{Z}}\big(\hat{F}_R(\mathcal{Z}), \hat{G}_R(\mathcal{Z})\big) & \text{(Theorem 3.5)} \\
\cong & \mathcal{Z} \otimes_R \mathrm{Hom}_{R\text{-}\mathrm{mod}\text{-}\mathcal{Z}}\big(\hat{F}_R(R), \hat{G}_R(R)\big) & \text{(Proposition 3.11 (d))} \\
\cong & \mathcal{Z} \otimes_R \mathrm{Hom}_{\mathrm{mod}\text{-}\mathcal{Z}}\big(\hat{F}_R(R), \hat{G}_R(R)\big) & \text{(Proposition 3.11 (a))} \\
\cong & \mathcal{Z} \otimes_R \mathrm{Hom}_{\mathrm{mod}\text{-}\mathcal{Z}}\big(\hat{F}_R \mathbb{V}_R \Delta_R(\lambda), \hat{G}_R \mathbb{V}_R \Delta_R(\lambda)\big) & \\
\cong & \mathcal{Z} \otimes_R \mathrm{Hom}_{\mathrm{mod}\text{-}\mathcal{Z}}\big(\mathbb{V}_R F_R \Delta_R(\lambda), \mathbb{V}_R G_R \Delta_R(\lambda)\big) & \\
\cong & \mathcal{Z} \otimes_R \mathrm{Hom}_{\mathfrak{g}_R}\big(F_R \Delta_R(\lambda), G_R \Delta_R(\lambda)\big) & \text{(Structure Theorem)} \\
\cong & \mathrm{Hom}_{\mathfrak{g}_{\mathcal{Z}}}\big(F_R \Delta_{\mathcal{Z}}(\lambda), G_R \Delta_{\mathcal{Z}}(\lambda)\big), &
\end{array}
$$

the latter by [Fie03, Proposition 2.4] and the tilting equivalence. Hence the Theorem follows for $T = S_{(0)}$. The case $T = \mathbb{C}$ follows then by specialisation or by copying the arguments. □

We call an endofunctor F of $\mathcal{O}_{T,\Lambda}$ *projective* if it is a direct sum of direct summands of some compositions of translation through walls. The following classification of projective functors (justifying their name) follows immediately:

Corollary 3.13 (The Classification Theorem). *Let $T = S_{(0)}$ or $T = \mathbb{C}$ and let $\mathcal{O}_{T,\Lambda}$ be an antidominant block with centre $\mathcal{Z}_T$. There are natural bijections of isomorphism classes*

$$
\begin{array}{c}
\big\{\textit{indecomposable projective functors on } \mathcal{O}_{T,\Lambda}\big\} \\
\updownarrow \textit{ 1:1} \\
\big\{\textit{indecomposable tilting objects of } \mathcal{O}_{T,\Lambda}\big\} \\
\updownarrow \textit{ 1:1} \\
\big\{\textit{indecomposable projective objects of } \mathcal{O}_{T,\tau(\Lambda)}\big\}
\end{array}
$$

Proof. The $F_T \Delta_{\mathcal{Z}_R}(\lambda)$'s for F a projective functor are tilting modules in $\mathcal{O}_{\mathcal{Z}_T,\Lambda}$ (via the tilting equivalence using [Fie03, Corollary 5.11, Proposition 2.4]). Any tilting module is obtained in such a way ([Fie04, Section 4]). Moreover, the isomorphism classes of (indecomposable) tilting objects in $\mathcal{O}_{\mathcal{Z}_T,\Lambda}$ correspond exactly to the ones in $\mathcal{O}_{S_{(0)},\Lambda}$ via specialisation, because $\mathcal{Z}$ is local. Hence the classification follows. □

We have a Krull-Remak-Schmidt property for projective functors:

Corollary 3.14. *Let $F : \mathcal{O}_{\mathbb{C},\Lambda} \to \mathcal{O}_{\mathbb{C},\Lambda}$ be a projective functor. Then*

a.) $F \cong \bigoplus_{i=1}^r F_i$ for some indecomposable functors F_i and $r \in \mathbb{N}$.

b.) Moreover, $F(\Delta_{\mathbb{C}}(\lambda)) \cong \bigoplus_{i=1}^r F_i(\Delta_{\mathbb{C}}(\lambda))$ is a decomposition into indecomposable direct summands.

Proof. The claim follows directly from the previous Theorem 3.12 and Corollary 3.13. □

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