SEMINAR: REPRESENTATION THEORY OF THE SYMMETRIC GROUP

1. Group representations

- Basics of the symmetric group
- For a general finite group: conjugacy classes (example: symmetric group), matrix representation/G-module
- Examples: trivial representation; sign and defining representation for the symmetric group, 1-dimensional complex representations for the cyclic group of order n, complex representations of an arbitrary commutative group
- Definition of the group algebra and the regular representation subrepresentation
- morphism, isomorphism
- irreducibility, indecomposability
- Trivial representation inside defining representation
- Tensor product, exterior product, symmetric powers, duals, Hom(V,W)

References. Sagan: 1.1-1.5, Examples 1.4.3 - 1.4.4
Fulton-Harris: pages 3-5, §3.4 (Definition of group algebra), Appendix B, Exercise 1.2 and Paragraph afterwards

2. Complete irreducibility, Maschke's theorem, Schur's Lemma

Consider complex representations (or over any other field of characteristic zero) of a finite group G

- any subrepresentation has a complement
- complete reducibility
- definition of kernel and image of a morphism
- Schur's Lemma
- Abelian groups and the symmetric group S_3
- Definition of character with examples
- Characters are class functions

References. Fulton-Harris: Prop. 1.5, Corollary 1.6 and paragraph afterwards, Lemma 1.7, Prop. 1.8, §1.3, §2.1 (not Prop. 2.1)
Sagan: 1.8.3, 1.8.4, 1.8.

3. Characters

Recall the definition of a character
Definition: Character table
Character of the direct sum, dual, tensor product, exterior product
Example: S_n
First projection formula and consequences (Orthogonality of characters of irreducible representations)

The number of irreducible representations equals the number of conjugacy classes
Examples: form S_4, S_5, or A_4

References. Fulton-Harris: Proposition 2.1, Example 2.6, choose from Exercise 2.2 - Exercise 2.7, Proposition 2.8, Theorem, 2.12, and Corollaries, Proposition 2.30
Sagan: 1.8-1.10

4. Example: S_n and constructing representations

Recall briefly the results from the previous talk which are needed
Example S_4
Question: How to construct representations in general?
Irreducible representations of $G_1 \times G_2$, Exterior powers, restriction
Induction (Definition and examples and the elementary approach of Sagan)
Frobenius reciprocity

References. Fulton-Harris: pages 18-19, Prop 3.12 (and all what needed to prove it), Proposition 3.17, Corollary 3.20, choose some examples from page 21-31 which you find interesting
Sagan: Theorem 1.11.3, Section 1.12, Theorem 1.12.6

5. Combinatorial tools / definition of Specht modules / Submodule Theorem/ Classification of simple modules

- Define the following notions (and illustrate them by examples!) partition, Ferres diagram, Young subgroup, Young tableau, λ-tabloid, permutation module, dominance ordering, lexicographic ordering
- Specht modules
- Examples
- The Submodule Theorem
- Classification of simple modules over a field of characteristic zero

References. :
Sagan: Sections 2.1-2.4
Other useful references: Fulton: 7.1 -7.2 James: Setions 1-4

6. Basis of Specht modules, branching rules

- Standard tableaux and a basis for the Specht modules
- Branching rule
- Decomposition of the permutation module M^μ
- Definition of Kostka numbers
- Young's rule

References. : Sagan: Prop. 2.5.9, Sections 2.8-2.9, Theorem 2.10.1 (only main idea of the proof), Section 2.11, in particular Theorem 2.11. Other useful references: James: Setion 9 Fulton: Section 7 (but the talk should follow roughly Sagan)

7. Generating functions / symmetric functions / Schur functions (Fundamental theorem of symmetric functions)

- Explain the idea of a generating function (following for example Sagan 4.1)
- Example: generating function for the number of partitions
- Symmetric functions (definition, basis of monomial symmetric functions, definition of p_λ, e_λ, h_λ)
- Generating function for elementary symmetric, complete homogeneous and power sum symmetric functions
- Main result: Fundamental theorem of symmetric functions
- Schur functions: definition, and basis (see e.g. Corollary 4.4.4 in Sagan)
- Mention different definitions of the Schur functions (e.g. Corollary 4.6.2 in Sagan, explain the ingredients and why these functions are symmetric, but do not give a full proof).

References. :
Sagan: Sections 4.1, 4.3-4.4, 4.6 other useful references:
Fulton: Section 6
MacDonald: Sections 1.1-1.3

8. Branching graph, Gelfand-Zetlin basis "Representation ring is isomorphic to ring of symmetric functions."

- Definition: Characteristic map (Sagan Section 4.7)
- Characteristic map is an isometry and an isomorphism of algebras (including the proof, but USING the statements of Section 4.6 with only giving an idea of the proof)
- Definition: Branching graph
- Definition: weight of an S_n-module
- Remark 2.15 in Kleshchev's book (see the introduction of the book for the notation)
- Explain the correspondence between weights and paths in $\mathbb{B}$

References. Sagan: Section 4.7
Alternatively: Fulton: Section 7.3 (but make sure you prepare all the ingredients)
Kleshchev: Introduction (for notation only, pages 9-11 and Remark 2.15)

9. Lie algebras of infinite dim matrices Fock space representations "Branching graph is a crystal graph."

- Definition of a Lie algebra (look it up yourself)
- Lecture 4 in Kac-Raina (leave out the physics terms and motivation except you are comfortable with it)
- Remark 2.3.13 of Kleshchev
- Draw parts of the branching graph

References. Kac-Raina: Lecture 4 (The focus of the talk should be on this)
Kleshchev: Remark 2.3.13

10. Sidetrack: Crystal bases and crystal graphs

Define the quantum group $Q_q(\mathfrak{sl}_2)$ for $\mathfrak{sl}_2$ (Jantzen, Section 1.1, with $k = \mathbb{Q}(q)$ $k = \mathbb{C}(q)$ and q just an indeterminant, see also Hong-Kang Example 4.2.6)

- Classification of irreducible (=simple) $Q_q(\mathfrak{sl}_2)$-modules (Jantzen Theorem 2.6, see also first part of Example 4.2.1 in Hong-Kang with $F = \mathbb{Q}$ or $F = \mathbb{C}$)
- Example 4.2.1 in Hong-Kang (Ignore $\mathcal{O}_int$, replace it by "finite dimensional modules")
- Definitions: crystal lattice, crystal basis, crystal graph
- Abstract definition of a crystal

References. :
Jantzen: Section 1.1, Theorem 2.6
Hong-Kang: 4.2.1-4.2.4, Theorem 4.2.5 and Example 4.2.6
Kleshchev: page 123 (figure out what $\mathfrak{g}$ is)

11. Summary and more explanations about the results so far

12. A few remarks on positive characteristic

Describe the basics on the representation theory of the symmetric group over fields of positive characteristic

- p-regular partitions
- Submodule theorem
- Specht modules
- crystal graph structure on the set of p-regular partitions

References. :
James: Section 10, Theorem 11.1, Lemma 11.3, Corollary 11.4, Theorem 11.5

13. Affine Kac Moody algebras and highest weight representations

The basic or fundamental representation of $\hat{\mathfrak{sl}}_n$

- The loop algebra
- the central extension of $\mathfrak{gl}_n$
- highest weight representation

References. Kac-Raina: Lecture 9
Kleshchev: Section 11
(Additional reference Grojnowski)