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COMPOSITION FACTORS OF QUOTIENTS OF THE UNIVERSAL ENVELOPING ALGEBRA BY PRIMITIVE IDEALS

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Abstract

Graded versions of the principal series modules of the category $\mathcal{O}$ of a semisimple complex Lie algebra $\mathfrak{g}$ are defined. Their combinatorial descriptions are given by some Kazhdan–Lusztig polynomials. A graded version of the Duflo–Zhelobenko four-term exact sequence is proved. This gives results about composition factors of quotients of the universal enveloping algebra of $\mathfrak{g}$ by primitive ideals; in particular an upper bound is obtained for the multiplicities of such composition factors. Explicit descriptions are given of principal series modules for Lie algebras of rank 2. It can be seen that these graded versions of principal series representations are neither rigid nor Koszul modules.

Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra of finite dimension and let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra. A longstanding open problem is to describe primitive ideals of $\mathcal{U}(\mathfrak{g})$. Although there is no classification of simple $\mathcal{U}(\mathfrak{g})$-modules, the primitive ideals are all given (see [**16**, 7.4]) by annihilators of simple objects inside the so-called category $\mathcal{O}$ which was introduced in [**7**]. For any weight λ there is a universal object in $\mathcal{O}$, the so-called *Verma module* with highest weight λ, denoted by $\Delta(\lambda)$. It is an object of the subcategory $\mathcal{O}_\lambda$ consisting of all objects with a fixed generalised central character χ_λ. Each Verma module $\Delta(\lambda)$ has a simple head, denoted by $L(\lambda)$. All simple objects in $\mathcal{O}$ are constructed in this way.

In general, there is not yet a satisfactory description of the annihilators of simple $\mathcal{U}(\mathfrak{g})$-modules. In the case when the simple module L in question is also a Verma module, the annihilator is given by a theorem of Duflo [**12**, 8.4.3]. In this special case even the composition factors of $\mathcal{U}(\mathfrak{g})/\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})} L$ can be computed by Kazhdan and Lusztig's [**20**] inductively defined polynomials. These polynomials can also be used to compute the number of distinct primitive ideals of $\mathcal{U}(\mathfrak{g})$ which are annihilators of simple modules in a fixed block.

This paper was motivated by the results and ideas of [**19**], where A. Joseph gives some indications on how a Jantzen filtration of the principal series modules should imply some results about composition factors of primitive quotients. Here the term 'primitive quotient' stands for a quotient of the form $\mathcal{U}(\mathfrak{g})/\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})} L$ for some simple $\mathcal{U}(\mathfrak{g})$-module L.

Instead of defining a Jantzen filtration of a principal series module we introduce a graded version of this module; we consider a (regular) integral block $\mathcal{O}_\lambda$ of $\mathcal{O}$ as a category of right modules over a finite dimensional algebra A (which is the

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endomorphism ring of a minimal projective generator of $\mathcal{O}_\lambda$). This algebra appears with a $\mathbb{Z}$-grading (see [**5**, Theorem 1.1.1]). Let $\overline{M}$ be an object of $\mathcal{O}_\lambda$ and let M be the corresponding right A-module. Then a graded version of $\overline{M}$ is a lift of M, that is, a graded right A-module $\tilde{M}$ such that $\tilde{M} \cong M$ when we forget the grading. For any simple module in $\mathcal{O}_\lambda$, there is a lift or graded version, concentrated in a single degree. Using a graded version of translation functors, as introduced in [**26**], it is possible to define graded versions of principal series modules. Since these modules are indecomposable, their lifts are unique up to isomorphism and grading shift [**5**, Lemma 2.5.3].

A combinatorial description of translation functors [**26**] gives rise to an explicit formula for the combinatorics of these graded versions of principal series modules in terms of elements of the Hecke algebra corresponding to the Weyl group of $\mathfrak{g}$ (Theorem 3.1). This combinatorial description gives as our main result an upper bound for the multiplicity of a simple module L in a given primitive quotient.

More precisely, given two elements x and y of the Weyl group we introduce a graded version $\mathcal{P}_{(x,y)}$ of the principal series module $\mathrm{Hom}_{\mathbb{C}}(\Delta(x\cdot 0), \nabla(y\cdot 0))^{\mathrm{adf}} \bigotimes_{\mathcal{U}} \Delta(0) \in \mathcal{O}_0$, where $\nabla(y\cdot 0)$ is the dual Verma module with simple socle $L(y\cdot 0)$. (For M any $\mathcal{U}(\mathfrak{g})$-bimodule, M^{adf} denotes the submodule consisting of 'finite vectors' for the adjoint action of $\mathfrak{g}$.)

On the other hand, we have a graded version of the simple composition factor $L(x\cdot 0)$, say $\tilde{L}(x\cdot 0)\langle i\rangle$, which is concentrated in degree i. In [**26**] we proved that there is an isomorphism (of $\mathbb{Z}[v, v^{-1}]$-modules) from the Hecke algebra to the Grothendieck group of gmof $-A$. We prove (Theorem 3.1) that $\mathcal{P}_{(w_o x, w_o y)}$ corresponds under this isomorphism to the Hecke algebra element $H_{x^{-1}}H_y$, a product of two (standard) basis elements such that

$$\big[\mathcal{P}_{(w_o x, w_o y)} : \tilde{L}(z\cdot 0)\langle i\rangle\big] = m_{x,y,z,i},$$

where $\sum_{i\in\mathbb{Z}} m_{x,y,z,i}v^i$ is the coefficient of some Kazhdan–Lusztig basis element D'_z which occurs in the expression of $H_{x^{-1}}H_y$ written in the Kazhdan–Lusztig basis. Therefore, these multiplicities can be (in principle) computed explicitly.

The primitive quotients (or rather their images in $\mathcal{O}$ under the equivalence between some category of Harish–Chandra bimodules and some subcategory of $\mathcal{O}$ described in [**6**]) are determined by homomorphisms (or intertwining maps) between the projective Verma module and some principal series representation. In our setup, these homomorphisms can be considered as homogeneous maps of graded modules. The combinatorics of both the Verma module and the principal series representation in question are, as we already mentioned, determined by some Kazhdan–Lusztig polynomials. Therefore we get an upper bound for the multiplicities of composition factors occurring in the images which describe the primitive quotients (Theorem 4.1):

$$\begin{aligned}\big[(\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(x\cdot 0)) \textstyle\bigotimes_{\mathcal{U}(\mathfrak{g})} \Delta(0) : L(y\cdot 0)\big]&\\ \leqslant \sum_{i\in\mathbb{Z}} \min\big\{\big[\mathcal{P}_{(w_o,w_o)} : \tilde{L}(y\cdot 0)\langle i\rangle\big], \big[\mathcal{P}_{(x,x)} : \tilde{L}(y\cdot 0)\langle i\rangle\big]\big\}&.\end{aligned}$$

This inequality becomes an equality for simple modules which do not occur with higher multiplicities in a dominant Verma module (Theorem 4.1 and Corollary 4.5). For higher multiplicities the situation is not that easy. At least we achieve a lower bound for the multiplicities, which can occur, so one can hope that the upper and the lower bounds coincide. Unfortunately this is not always the case: a counter-example can be found for type B_3 (see [**23**]). Nevertheless, we are able to prove a graded version of the Duflo–Zhelobenko four-step exact sequence (4.2) which implies an even stronger (but also more technical) result than the formula above (Theorem 4.3). Roughly speaking, it states that the composition factors of $\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(x\cdot 0)\bigotimes_{\mathcal{U}(\mathfrak{g})}\Delta(0)$ are exactly those which occur in $\mathcal{P}_{(w_o,w_o)}$ and in $\mathcal{P}_{(x,x)}$ in the same degree (up to a certain permutation for higher multiplicities!).

In the last section we prove a general statement about translation functors and Koszul modules. It turns out that graded versions of principal series modules are neither rigid nor Koszul in general.

1. *Principal series modules and graded category $\mathcal{O}$*

1.1. *The category $\mathcal{O}$*

Let $\mathfrak{g}\supset\mathfrak{b}\supset\mathfrak{h}$ be a semisimple complex Lie algebra with a chosen Borel and a fixed Cartan subalgebra. Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{b}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}$ be the corresponding Cartan decomposition. The corresponding universal enveloping algebras are denoted by $\mathcal{U}=\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{b})$ etc.

We consider the category $\mathcal{O}$ which is the full subcategory of the category of $\mathcal{U}$-modules whose objects are

(1) finitely generated;
(2) locally finite for $\mathfrak{n}$;
(3) acted on diagonally by $\mathfrak{h}$.

Standard references are [**7**, **15**, **16**].

We denote by W the Weyl group with longest element w_o. Let ρ denote the half-sum of positive roots. The 'translated' (or 'dot') action of W on $\mathfrak{h}^*$ is defined by $x\cdot\lambda=x(\lambda+\rho)-\rho$. The action of the centre $\mathcal{Z}=\mathcal{Z}(\mathcal{U})$ of $\mathcal{U}$ decomposes the category into direct summands indexed by maximal ideals of $\mathcal{Z}$:

$$\mathcal{O}=\bigoplus_{\chi\in\operatorname{Max}\mathcal{Z}}\mathcal{O}_\chi=\bigoplus_{\lambda\in\mathfrak{h}^*/(W\cdot)}\mathcal{O}_\lambda, \tag{1.1}$$

where $\mathcal{O}_\chi$ denotes the subcategory of $\mathcal{O}$ consisting of all objects annihilated by some power of χ. It denotes the same subcategory as $\mathcal{O}_\lambda$ if $\xi(\lambda)=\chi$, where ξ denotes the Harish–Chandra isomorphism. Here, the second sum runs over orbits of the translated action of W on $\mathfrak{h}^*$. The summand $\mathcal{O}_\lambda$ is called *integral* if λ is integral, that is, if $\langle\lambda+\rho,\check{\alpha}\rangle\in\mathbb{Z}$ for any coroot $\check{\alpha}$. We denote by $W_\lambda=\{w\in W\mid w\cdot\lambda=\lambda\}$ the stabiliser of λ in W and call λ (and $\mathcal{O}_\lambda$ respectively) regular if W_λ is trivial.

For all $\lambda\in\mathfrak{h}^*$ we have a standard module, the Verma module $\Delta(\lambda)=\mathcal{U}\bigotimes_{\mathcal{U}(\mathfrak{b})}\mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ denotes the irreducible $\mathfrak{h}$-module with weight λ enlarged by the trivial action of $\mathfrak{n}$ to a module over the Borel subalgebra. This Verma module is a highest weight module of highest weight λ and has central character $\xi(\lambda)$. We denote by $L(\lambda)$ the unique irreducible quotient of $\Delta(\lambda)$. We denote by τ the Chevalley antiautomorphism (see [**16**, 2.1]). Let $\star$ denote the duality of $\mathcal{O}$, that is $M^\star$ is the maximal $\mathfrak{h}$-semisimple submodule of the contragredient representation M^* with the τ-twisted action, that is $(xf)(m)=f(\tau(x)m)$ for all $x\in\mathfrak{g}$, $f\in M^*$ and $m\in M$. We denote by $\nabla(\lambda)$ the dual Verma module $\Delta(\lambda)^\star$.

Let λ, $\mu \in \mathfrak{h}^*$ lying in the same Weyl chamber be such that $\mu - \lambda$ is integral. The translation functor from $\mathcal{O}_\lambda$ to $\mathcal{O}_\mu$ is the functor

$$\begin{aligned} \theta_\lambda^\mu : \quad & \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\mu \\ & M \longmapsto \mathrm{pr}_\mu(M \otimes E(\mu - \lambda)), \end{aligned}$$

where pr_μ is the projection onto $\mathcal{O}_\mu$ and $E(\mu - \lambda)$ is the finite dimensional simple $\mathfrak{g}$-module with extremal weight $\mu - \lambda$. Let s be a simple reflection, suppose that $W_\mu = \{1, s\}$, then translation *through* the s-wall is the composition of functors $\theta_s = \theta_\mu^\lambda \circ \theta_\lambda^\mu$. For more details concerning these functors see [**15**, **16**].

1.2. *Gradings*

For any graded ring R we denote by (g) mof $-R$ the category of finitely generated (graded) right R-modules. For a regular integral block, say $\mathcal{O}_0$, of $\mathcal{O}$, let A denote the endomorphism ring of a minimal projective generator P. In [**5**, Theorem 1.1.3] it is explained how this ring becomes a $\mathbb{Z}$-graded ring. In the following, we use the term 'graded' instead of '$\mathbb{Z}$-graded'. Let f be the grading-forgetting functor. For $m \in \mathbb{Z}$ let $M\langle m\rangle$ be the graded module defined by $M\langle m\rangle_n := M_{n-m}$ with the same module structure as M, that is $\mathrm{f}(M\langle m\rangle) = \mathrm{f}(M)$.

Definition 1.1. Let B and D be graded rings. We call a module $M \in \mathrm{mof}-B$ *gradable* if there exists a graded module $\tilde{M} \in \mathrm{mof}-B$ such that $\mathrm{f}(\tilde{M}) \cong M$. In this case, the module $\tilde{M}$ is a *lift* of M.

An object $M \in \mathcal{O}_0$ is *gradable* if $\mathrm{Hom}_{\mathfrak{g}}(P, M)$ is a gradable A-module, where $A = \mathrm{End}_{\mathfrak{g}}(P)$ is graded as described in [**5**, Theorem 1.1.4] or [**26**, Theorem 2.1]. By abuse of language, a lift of $\mathrm{Hom}_{\mathfrak{g}}(P, M)$ is often called a *lift of* M. Let B and D be graded rings. We call a functor $F : \mathrm{mof}-B \longrightarrow \mathrm{mof}-D$ *gradable* if there exists a functor of graded categories $\tilde{F} : \mathrm{gmof}-B \longrightarrow \mathrm{gmof}-D$ (in the sense of [**2**, Appendix E.3]) which induces F after forgetting the grading. If there is such a functor $\tilde{F}$, we call it a *lift* of F. We call a functor F on $\mathcal{O}_0$ gradable if it induces a gradable functor on mof $-A$.

We recall two well-known facts concerning graded modules which will be used several times in the following.

Lemma 1.2. *Let M, $N \in \mathcal{O}_0$ be gradable with lifts $\tilde{M}$ and $\tilde{N} \in \mathrm{gmof}-A$, respectively.*

(i) *If M is indecomposable, then $\tilde{M}$ is uniquely defined up to isomorphism and grading shift, that is, for any $\hat{M} \in \mathrm{gmof}-A$ such that $\mathrm{f}\,\hat{M} \cong M$, there exists an isomorphism of graded modules $\tilde{M} \cong \hat{M}\langle i\rangle$ for some $i \in \mathbb{Z}$.*

(ii) *Let $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{mof}-A}(M, N) \leqslant 1$; then any morphism $f \in \mathrm{Hom}_{\mathrm{mof}-A}(\tilde{M}, \tilde{N})$ is a graded homomorphism, homogeneous of some degree $i \in \mathbb{Z}$.*

Proof. For the first statement see [**5**, Lemma 2.5.3]. Since we have naturally $\mathrm{Hom}_{\mathrm{mof}-A}(M, N) = \mathrm{Hom}_{\mathrm{mof}-A}(\tilde{M}, \tilde{N}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{gmof}-A}(\tilde{M}, \tilde{N}\langle i\rangle)$, it follows from our assumption that $\mathrm{Hom}_{\mathrm{mof}-A}(M, N) = \mathrm{Hom}_{\mathrm{gmof}-A}(\tilde{M}, \tilde{N}\langle i\rangle)$ for some $i \in \mathbb{Z}$. The second statement of the lemma follows. □

In [**5**] and also in [**26**, Lemma 3.2] it is proved that every simple module in $\mathcal{O}_0$ is gradable. Every lift is concentrated in a single degree. All Verma modules in $\mathcal{O}_0$ are gradable. Their standard lifts defined in [**5**] or [**26**, Theorem 3.6] are denoted by $\tilde{\Delta}(x\cdot 0)$. In [**26**, 3.2] a graded lift $\tilde{\theta}_s$ of the translation through the wall was introduced. It has the property that for any graded module $M \in \mathrm{gmof}-A$ the canonical maps $\mathrm{adj}_1(M): M \longrightarrow \tilde{\theta}_s M$ and $\mathrm{adj}_2(M): \tilde{\theta}_s M \longrightarrow M$ are homogeneous of degree 1. This provides a combinatorial description in terms of the Hecke algebra.

1.3. *Combinatorial description of graded translation functors*

We denote the Grothendieck group $[\mathrm{gmof}-A]$ of $\mathrm{gmof}-A$ by $[\mathcal{O}_0^{\mathbb{Z}}]$ to indicate that we are in fact interested in $\mathcal{O}_0$. A basis of this group is given by the isomorphism classes of the graded lifts $\tilde{\Delta}(x\cdot 0)\langle n\rangle$ of Verma modules in $\mathcal{O}_0$, where $n \in \mathbb{Z}$ and $x \in W$. Moreover, $[\mathcal{O}_0^{\mathbb{Z}}]$ can be considered as a $\mathbb{Z}[v,v^{-1}]$-module defined by $v^n[M] := [M\langle n\rangle]$. As a $\mathbb{Z}[v,v^{-1}]$-module it is isomorphic to the Hecke algebra H of W. This is by definition (see [**22**] and [**8**, IV, 2, Ex. 22]) the free $\mathbb{Z}[v,v^{-1}]$-module with basis $\{H_x \mid x \in W\}$ subject to the relations

$$\begin{aligned} H_s^2 &= H_e + (v^{-1} - v)H_s && \text{for a simple reflection } s,\\ H_x H_y &= H_{xy} && \text{if } l(x) + l(y) = l(xy). \end{aligned}$$

The main result in [**26**] is that the graded version $\tilde{\theta}_s$ of the translation through the s-wall satisfies the following combinatorial description given by the following commuting diagram.

$$\begin{array}{ccc} H & \xrightarrow{v^n H_x \longmapsto [\tilde{\Delta}(x\cdot 0)\langle n\rangle]} & [\mathcal{O}_0^{\mathbb{Z}}] \\ {\scriptstyle \cdot(H_s+v)}\downarrow & & \downarrow{\scriptstyle [\tilde{\theta}_s]} \\ H & \xrightarrow{v^n H_x \longmapsto [\tilde{\Delta}(x\cdot 0)\langle n\rangle]} & [\mathcal{O}_0^{\mathbb{Z}}] \end{array} \tag{1.2}$$

In [**26**] it is also explained how to define a graded version of duality. This is a functor d on $\mathrm{gmof}-A$ which commutes with graded translation through the wall and which becomes the usual duality, coming from the duality on $\mathcal{O}_0$, after forgetting the grading. Moreover, for a simple module L in degree 0, we have $d(L\langle n\rangle) \cong L\langle -n\rangle$. In the graded Grothendieck group this duality is therefore described as the identity on isomorphism classes of simple modules in degree 0 and the rule $v \longmapsto v^{-1}$.

Kazhdan and Lusztig defined an involutive automorphism (or duality) $H \longmapsto \overline{H}$ on H with the property $H_x \longmapsto (H_{x^{-1}})^{-1}$ and $vH_e \longmapsto v^{-1}H_e$. A crucial result of [**20**] is the existence of a self-dual Kazhdan–Lusztig basis, that is, a basis $\{\underline{H}_x \mid x \in W\}$ uniquely defined by the properties $\overline{\underline{H}_x} = \underline{H}_x$ and $\underline{H}_x \in H_x + \sum_{y \neq x} v\mathbb{Z}[v]H_y$. The (in the meantime proved) Kazhdan–Lusztig conjectures [**20**] and the results of [**5**] imply that this duality corresponds to the duality d in diagram (1.2). It is perhaps worth mentioning that the commutativity of the diagram above and the definition of d do not require the truth of the Kazhdan–Lusztig conjectures, which, however, are needed to describe the Hecke algebra elements corresponding to isomorphism classes of simple and projective objects.

For M, $L \in \mathrm{gmof}-A$ (or M, $L \in \mathrm{mof}-A$ respectively), where L is a simple object, we denote by $[M:L]$ the multiplicity how often L occurs as a composition factor in M. In other words, $[M] = \sum [M:L][L]$ in the corresponding Grothendieck group, where the sum runs over isomorphism classes of simple objects. Let $\tilde{L}(x\cdot 0)$

denote the (standard) lift of the simple module $L(x\cdot 0)$, concentrated in degree zero. The following multiplicity formula holds for any simple reflection s such that $xs > x$ (see [**26**, Theorem 5.1]):

$$[\tilde{\theta}_s\tilde{L}(xs\cdot 0):\tilde{L}(xs\cdot 0)\langle j\rangle] = \begin{cases} 1 & \text{if } j=\pm 1,\\ 0 & \text{otherwise.}\end{cases} \tag{1.3}$$

Here, the simple module $\tilde{L}(xs\cdot 0)\langle 1\rangle$ is a submodule and $\tilde{L}(xs\cdot 0)\langle -1\rangle$ is a quotient of $\tilde{\theta}_s\tilde{L}(xs\cdot 0)$.

1.4. *Principal series modules*

Recall that for a $\mathcal{U}$-bimodule X the *adjoint action* of $\mathfrak{g}$ is defined by $g.x = gx - xg$ for $g\in\mathfrak{g}$ and $x\in X$. For M and $N\in\mathcal{O}$ the set $\mathrm{Hom}_{\mathbb{C}}(M,N)$ is a $\mathcal{U}$-bimodule in a natural way. The subspace $\mathcal{L}(M,N)$, consisting of vectors lying in a finite dimensional subspace which is invariant under the adjoint action of $\mathfrak{g}$, is an object of the category of Harish–Chandra bimodules (see [**6**, II, **16**, Kapitel 6]). Let x, $y\in W$. Then the $\mathcal{U}$-module $\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0)$ is an object of $\mathcal{O}_0$ via the Bernstein–Gelfand equivalence [**6**, 5.9]. These are the *principal series modules.*

These principal series modules have the following properties.

(A) (See [**18**, Corollary 2.9 and Lemma 2.10] and [**17**, Lemma 2.5].) For all $x\in W$ there are isomorphisms of $\mathcal{U}$-bimodules:

$$\mathcal{L}(\Delta(x\cdot 0),\nabla(w_o\cdot 0))\cong\mathcal{L}(\Delta(0),\Delta(x^{-1}w_o\cdot 0)),$$
$$\mathcal{L}(\Delta(x\cdot 0),\nabla(0))\cong\mathcal{L}(\Delta(0),\nabla(x^{-1}\cdot 0)).$$

In the Grothendieck group of $\mathcal{O}_0$ the equality

$$[\mathcal{L}(\Delta(x^{-1}\cdot 0),\nabla(y\cdot 0))\bigotimes\nolimits_{\mathcal{U}}\Delta(0)] = [\Delta(xy\cdot 0)]$$

holds for all x, $y\in W$ (by Frobenius reciprocity; see remark [**18**, 3.1] or for a proof [**11**, 9.6.2]).

(B) All principal series modules are indecomposable [**25**].

In particular, all Verma modules and dual Verma modules in $\mathcal{O}_0$ are examples of principal series modules.

2. *Principal series modules as gradable objects*

In the following section we show that all principal series representations are gradable and we give a combinatorial description in terms of basis elements of the Hecke algebra.

THEOREM 2.1. *For x, $y\in W$ the module $\mathcal{L}(\Delta(x^{-1}\cdot 0),\nabla(y\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0)\in\mathcal{O}_0$ is gradable. A lift is unique up to isomorphism and grading shift.*

Proof. The short exact sequence in $\mathcal{O}_0$

$$\nabla(y\cdot 0)\hookrightarrow\theta_s\nabla(y\cdot 0)\twoheadrightarrow\nabla(ys\cdot 0)$$

with $ys<y$ gives an exact sequence

$$0\longrightarrow\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0))\xrightarrow{f}\mathcal{L}(\Delta(x\cdot 0),\theta_s\nabla(y\cdot 0))\xrightarrow{g}\mathcal{L}(\Delta(x\cdot 0),\nabla(ys\cdot 0)) \tag{2.1}$$

of Harish–Chandra bimodules with trivial central character from the right. Therefore we get (by [**16**, 6.33 (6)]) the following short exact sequence in $\mathcal{O}_0$:

$$\begin{aligned} 0 \longrightarrow \mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0) &\overset{f}{\hookrightarrow} \theta_s(\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0)) \\ \overset{g}{\twoheadrightarrow} \mathcal{L}(\Delta(x\cdot 0),\nabla(ys\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0) &\longrightarrow 0, \end{aligned} \tag{2.2}$$

if we can show that g is surjective. By property (A) in Subsection 1.4 we get $[\theta_s\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0)] = [\theta_s\Delta(x^{-1}y\cdot 0)] = [\Delta(x^{-1}y\cdot 0)]+[\Delta(x^{-1}ys\cdot 0)] = [\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0)]+[\mathcal{L}(\Delta(x\cdot 0),\nabla(ys\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0)]$; hence the morphism g has to be surjective. (Perhaps more conceptual is to say that $\mathcal{L}(\Delta(x\cdot 0),\bullet)$ is exact on modules having a filtration with subquotients isomorphic to dual Verma modules, since it commutes with tensoring with finite dimensional (left) $\mathfrak{g}$-modules and $\nabla(0)$ is injective.)

If $y = w_o$ holds, the second term in (2.2) is by property (A) isomorphic to

$$\mathcal{L}(\Delta(0),\Delta(x^{-1}w_o\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0) \cong \Delta(x^{-1}w_o\cdot 0)$$

and is therefore gradable for all $x\in W$. Since translation through the wall has a graded lift, the translated module is also gradable. Since the homomorphism space

$$\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0)),\mathcal{L}(\Delta(x\cdot 0),\theta_s\nabla(y\cdot 0)))$$

is one-dimensional [**26**, Endomorphism Theorem] for all x and $y\in W$, the homomorphism f has to be homogeneous considered as a map between graded modules (Lemma 1.2), so the cokernel in (2.2) is gradable. Inductively the gradability of all principal series modules follows. The uniqueness of the lifts follows from Lemma 1.2 using property (B). □

We choose a lift of principal series modules, such that the surjection in (2.2) is homogeneous of degree zero. This does not depend on the reduced expression for x, because the lift of the simple module corresponding to the longest element occurs in the same degree for any chosen reduced expression by formula (1.3). We denote the so-defined lift of $\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0))\bigotimes_{\mathcal{U}}\Delta(0)$ by $\mathcal{P}_{(x,y)}$. If $x = y$ we just write $\mathcal{P}_x$.

For Verma modules, these lifts coincide by definition with the lifts defined in [**26**, Theorem 3.6] and therefore also with the lifts in [**5**, Proposition 3.5.7].

We get the following short exact sequences of graded modules.

Theorem 2.2. *Let $y\in W$ and let s be a simple reflection satisfying $ys > y$. There are short exact sequences of graded modules*

$$\begin{aligned} \mathcal{P}_{(x,ys)}\langle 1\rangle \overset{\mathrm{adj}_1}{\hookrightarrow} \tilde{\theta}_s\mathcal{P}_{(x,ys)} \twoheadrightarrow \mathcal{P}_{(x,y)}, \\ \mathcal{P}_{(x,ys)} \hookrightarrow \tilde{\theta}_s\mathcal{P}_{(x,y)} \overset{\mathrm{adj}_2}{\twoheadrightarrow} \mathcal{P}_{(x,y)}\langle -1\rangle. \end{aligned}$$

Proof. The first sequence follows from the fact that the canonical map $\mathrm{adj}_1(M)$ is of degree 1 for all graded modules M and from the convention for the lifts of the quotients.

Since the adjunction morphism adj_2 is homogeneous of degree 1, the surjection

$$\tilde{\theta}_s\mathcal{P}_{(x,y)}\twoheadrightarrow\mathcal{P}_{(x,y)}\langle -1\rangle$$

is homogeneous of degree zero. The simple module $\Delta(w_o \cdot 0)$ occurs in $\mathcal{P}_{(x,y)}$ as a composition factor with multiplicity one. Let i be the degree in which it is concentrated. Thus it occurs twice in $\tilde{\theta}_s\mathcal{P}_{(x,y)}$, namely in degrees $i+1$ and $i-1$ (see (1.3)). Comparing this with the first sequence it follows that $\tilde{\theta}_s\mathcal{P}_{(x,ys)}$ has this composition factor exactly in degree $i+2$ and in degree i, so it has to appear in $\mathcal{P}_{(x,ys)}$ in degree $i+1$. Hence we have proved that the injection of the second sequence must be homogeneous of degree zero, and so we are done. □

Corollary 2.3. *With the same assumptions as in the theorem we get an isomorphism of graded modules*

$$\tilde{\theta}_s\mathcal{P}_{(x,y)} \cong \tilde{\theta}_s\mathcal{P}_{(x,ys)}\langle -1\rangle.$$

Proof. Note that there is an isomorphism after forgetting the grading. Property (B) in Subsection 1.4 and Lemma 1.2 imply the existence of an isomorphism of graded modules $\tilde{\theta}_s\mathcal{P}_{(x,y)} \cong \tilde{\theta}_s\mathcal{P}_{(x,ys)}\langle j\rangle$ for some $j \in \mathbb{Z}$. Comparing the two sequences in Theorem 2.2 gives $j = -1$. □

3. *The combinatorics of principal series modules*

The combinatorial description of translation functors provides a combinatorial description of the graded lifts of principal series modules.

Theorem 3.1. *Under the isomorphism in* (1.2), *the element*

$$\left[\mathcal{P}_{(w_o x, w_o y)}\right] \in \left[\mathcal{O}_0^{\mathbb{Z}}\right]$$

corresponds to the following element of the Hecke algebra:

$$H_{x^{-1}}H_y.$$

Proof. For $y=e$, the isomorphism class of the graded module $\mathcal{P}_{(w_o x, w_o)} \cong \mathcal{P}_{(e,x^{-1})}$, corresponds to $H_{x^{-1}} = H_{x^{-1}}H_e$ by definition of the isomorphism in (1.2). Let $y \in W$ and let s be a simple reflection such that $ys > y$ holds. Theorem 2.2 gives a short exact sequence of graded modules of the form

$$0 \longrightarrow \mathcal{P}_{(w_o x, w_o y)}\langle 1\rangle \longrightarrow \tilde{\theta}_s\mathcal{P}_{(w_o x, w_o y)} \longrightarrow \mathcal{P}_{(w_o x, w_o ys)} \longrightarrow 0.$$

Assuming that the assertion is true for y, the isomorphism class of $\mathcal{P}_{(w_o x, w_o ys)}$ corresponds to the element

$$(H_{x^{-1}}H_y)(H_s+v) - (vH_{x^{-1}}H_y) = H_{x^{-1}}H_yH_s = H_{x^{-1}}H_{ys}$$

of the Hecke algebra. This is just what we had to show. □

Remark 3.2. (a) From the relation $H_s^2 = H_e + (v^{-1}-v)H_s$ it follows inductively that the simple module corresponding to the dominant weight occurs in $\mathcal{P}_x$ always in degree zero.

(b) The Bernstein–Gelfand equivalence and the interchanging of right and left module structures for Harish–Chandra bimodules gives also an equivalence of categories between $\mathcal{O}_0$ and the category of Harish–Chandra bimodules with trivial central character from the left-hand side. We can also define a graded version of the translation functor through the wall (on the right-hand side) which implies an analogous diagram as in (1.2). It is then straightforward to see that it defines the same combinatorics for principal series modules as described in Theorem 3.1. Details can be found in [**23**].

The character formulae (A) of Subsection 1.4 can now be deduced purely combinatorially (without using Kazhdan–Lusztig conjectures).

COROLLARY 3.3. *All principal series modules $\mathcal{P}_x$ define after forgetting the grading the same elements in the Grothendieck group.*

Proof. For any simple reflection s the equality $H_s^2 = H_e + (v^{-1} - v)H_s$ holds. Evaluating it at the point 1 yields the same composition factors of $\mathcal{P}_{w_o s}$ as for the projective Verma module. Let $x \in \mathcal{W}$ and let s be a simple reflection such that $x = sy$ with $l(y) = l(x) - 1$ holds. The equalities

$$\begin{aligned} H_{x^{-1}} H_x &= H_{y^{-1}} H_s^2 H_y \\ &= H_{y^{-1}} H_e H_y + (v^{-1} - v) H_{y^{-1}} H_s H_y \end{aligned}$$

show that the composition factors (with multiplicities) of $\mathcal{P}_x$ and of $\mathcal{P}_y$ coincide after forgetting the grading. Induction gives the desired result. □

The motivation for the last result of this section was given by [**19**] where it is proved that H_e gives the gradation dual to the one described by $H_{w_o} H_{w_o}$.

THEOREM 3.4. *There is an isomorphism of graded modules $\mathcal{P}_{(x,y)} \cong d\mathcal{P}_{(w_o x, w_o y)}$ for all x, $y \in W$.*

Proof. By the duality theorem ([**1**, 3.4 and Corollary 2.1] or [**25**, Theorem 3.1]), there is an isomorphism after forgetting the grading. Since the principal series modules are indecomposable, their lifts are unique up to isomorphism and grading shift (see Lemma 1.2). Since the composition factor corresponding to the dominant weight occurs in any $\mathcal{P}_x$ in degree zero, it also occurs in degree zero in $d\mathcal{P}_x$ by definition of d. This proves the claim. (We can also mimic the proof of the non-graded duality theorem with graded exact sequences from Theorem 2.2.) □

COROLLARY 3.5. *For $x, y \in \mathcal{W}$ the equality*

$$\left[\mathcal{P}_{(x,y)}\right] = \left[d\mathcal{P}_{(w_o x, w_o y)}\right] \tag{3.1}$$

holds in the graded Grothendieck group of $\mathcal{O}_0$.

Proof. This is in immediate consequence of the previous theorem. Nevertheless, let us give an idea how to prove it using the conjectures and combinatorics of Kazhdan and Lusztig. In the notation of [**22**], the self-dual element $\underline{\tilde{H}}_y$ corresponds to C_y in [**20**]. The formula [**21**, (5.1.8)], that is $D'_x = C_{x w_o} H_{w_o}$, shows that $\underline{\tilde{H}}_y H_{w_o}$ corresponds to $[L(y \cdot 0)]$ under the isomorphism (1.2). For x, $y \in W$ we define Laurent polynomials h_y by

$$H_{x^{-1}} H_y = \sum_{z \in W} h_z (\underline{\tilde{H}}_{z w_o} H_{w_o}).$$

Dualising and multiplying by $H_{w_o}^2$ on the right-hand side gives the formula

$$\sum_y \overline{h}_y \underline{\tilde{H}}_{yw_o} H_{w_o} = \overline{H}_{x^{-1}} \overline{H}_y H_{w_o}^2.$$

An easy computation shows that $H_{w_o}^2$ is central in H; hence

$$\overline{H}_{x^{-1}} \overline{H}_y H_{w_o}^2 = (H_x)^{-1} H_{w_o}^2 (H_{y^{-1}})^{-1} = H_{x^{-1}w_o} H_{w_o y} = H_{(w_o x)^{-1}} H_{w_o y}.$$

Since the polynomials h_y decode the multiplicities of the graded composition factors, the assertion follows. □

4. *Primitive quotients and their composition factors*

We consider $\mathrm{Ann}_{\mathcal{U}} L(x\cdot 0)$, the annihilator of the simple module $L(x\cdot 0)\in\mathcal{O}_0$. By a theorem of Duflo [**12**, Proposition 10] the corresponding primitive quotient can be described as the image of some homomorphism between certain principal series modules. More precisely, in our setup it means that there is a morphism

$$\psi_x : \Delta(0) \longrightarrow \mathcal{L}(\Delta(x), \nabla(x)) \bigotimes\nolimits_{\mathcal{U}} \Delta(0) \tag{4.1}$$

such that $\mathrm{im}\,\psi_x \cong \mathcal{U}/\mathrm{Ann}_{\mathcal{U}} L(x) \bigotimes_{\mathcal{U}} \Delta(0)$. Therefore, the image of the *Duflo map* ψ_x corresponds to the primitive quotient $\mathcal{U}/\mathrm{Ann}_{\mathcal{U}} L(x)$ via the Bernstein–Gelfand equivalence of categories [**6**].

The following theorem confirms the ideas and conjectures of A. Joseph [**19**].

Theorem 4.1 (upper bound for composition factors). *For all $x\in W$, the (unique up to a scalar non-trivial) morphism*

$$\psi_x : \mathcal{P}_{w_o} \longrightarrow \mathcal{P}_x$$

is homogeneous of degree zero.

In particular, there is an upper bound for the multiplicities

$$[(\mathcal{U}/\mathrm{Ann}\, L(x\cdot 0)) \bigotimes\nolimits_{\mathcal{U}} \Delta(0) : L(y\cdot 0)] \leqslant \sum_{i\in\mathbb{Z}} \min\{[\mathcal{P}_{w_o} : \tilde{L}(y\cdot 0)\langle i\rangle], [\mathcal{P}_x : \tilde{L}(y\cdot 0)\langle i\rangle]\}.$$

Remark 4.2. Using the combinatorial description of Theorem 3.1, these upper bounds for the multiplicities can be computed explicitly.

Proof of Theorem 4.1. All principal series modules $\mathcal{P}_x$ (with $x\in W$) have the same non-graded characters because of property (A) in Subsection 1.4. Therefore, the homomorphism space between $\Delta(0)$ and $\mathcal{L}(\Delta(x\cdot 0), \nabla(x\cdot 0)) \bigotimes_{\mathcal{U}} \Delta(0)$ is one-dimensional for all x, since $\Delta(0)\in\mathcal{O}_0$ is projective. Hence (by Lemma 1.2) the map ψ_x between their graded lifts is homogeneous. On the other hand, the simple module $L(0)$ is always in the image of each Duflo map, because its annihilator is maximal in the set of all primitive ideals with the inclusion ordering (see [**16**, Corollary 7.2]). By Theorem 3.1 and Remark 3.2(a) the lift of this simple module occurs in each $\mathcal{P}_x$ in degree zero. Therefore, ψ_x is homogeneous of degree zero. Hence, the statement follows immediately from the result (4.1) of Duflo. □

It turns out (see Corollary 4.5) that the previous theorem provides all the information for a simple module occurring with single multiplicity in the dominant

Verma module $\Delta(0)$. In particular, the result gives all composition factors of primitive quotients for Lie algebras of rank at most 2 in a purely combinatorial way. For type A_3 with a given involution x, the composition factors of $\operatorname{im}\psi_x$ are (with multiplicities!) just the ones which occur in $\mathcal{P}_{w_o}$ and in $\mathcal{P}_x$ in the same degree. However, it is not the case in general. Detailed examples can be found in [**23**].

We need some tool to 'distinguish' composition factors in the case of higher multiplicities. This is (as suggested in [**19**]) somehow given by the following graded version of the four-step exact sequence.

THEOREM 4.3 (graded Duflo–Zhelobenko sequence). (i) *Let $x \in W$ and let s be a simple reflection such that $sx > x$. There is a short exact sequence of graded modules*

$$0 \longrightarrow \mathcal{P}_{(x,sx)}\langle 1\rangle \longrightarrow \mathcal{P}_{(sx,sx)} \xrightarrow{f_{sx,x}} \mathcal{P}_{(x,x)} \longrightarrow \mathcal{P}_{(x,sx)}\langle -1\rangle \longrightarrow 0. \tag{4.2}$$

(ii) *Moreover, the homomorphism space between two consecutive modules in the sequence is one-dimensional. In particular, after forgetting the grading, this is just the Duflo–Zhelobenko exact sequence from* [**19**].

Proof. The existence of such a sequence in the non-graded case is given by the Duflo–Zhelobenko sequence [**19**]. Since all the homomorphism spaces in question are one-dimensional [**25**, Endomorphism Theorem], all maps are homogeneous (Lemma 1.2). The only thing one has to check is the degree of these maps. The map in the middle has to be of degree zero, since in its image the finite-dimensional simple module appears as composition factor and its graded lift always occurs in degree zero in all principal series modules $\mathcal{P}_x$ (see Remark 3.2(a)). The others are given by an easy calculation inside the Hecke algebra using Theorem 3.1. □

The graded four-step exact sequence gives us a lower bound for multiplicities for composition factors of primitive quotients as follows. Let L be a simple module in $\operatorname{gmof}-A$ concentrated in degree zero. We denote by $m_{L,x}$ the maximal integer i such that $[\mathcal{P}_x : L\langle i\rangle] \neq 0$ and set $m_{L,x} = -\infty$ if $[\mathrm{f}\,\mathcal{P}_x : \mathrm{f}\,L] = 0$. The main result is given by the following.

THEOREM 4.4 (lower bounds for composition factors). (i) *Let $x \in W$ with reduced expression $\hat{x}$. Let $s_1 \dots s_r\hat{x}$ be a reduced expression of w_o. Then the Duflo map ψ_x is up to a non-zero scalar the composition $f_{s_r x,x} \circ f_{s_{r-1}s_r x, s_r x} \circ \dots \circ f_{s_1 w_o, s_2 s_1 w_o} \circ f_{w_o, s_1 w_o}$ of maps arising in the four-term exact sequences* (4.2).

(ii) *Let $x \in W$ and s be a simple reflection such that $sx > x$. Let $L \in \operatorname{gmof}-A$ be a simple object. Then*

$$[\operatorname{im} f_{sx,x} : L\langle j\rangle] = 0, \qquad \text{if } j > m_{L,sx} \text{ or } j \not\equiv m_{L,sx} \bmod 2,$$

and for any $k \in \mathbb{Z}$,

$$\begin{aligned}[\operatorname{im} f_{sx,x} : L\langle m_{L,x} - 2k\rangle] = [\mathcal{P}_x : L\langle m_{L,x} - 2k\rangle] + \sum_{j=0}^{k-1}([\mathcal{P}_x : L\langle m_{L,x} - 2j\rangle] \\ - [\mathcal{P}_{sx} : L\langle m_{L,x} - 2j\rangle]).\end{aligned} \tag{4.3}$$

(iii) *Moreover, if* $j > m_{L,w_o}$ *or* $j \not\equiv m_{L,w_o} \bmod 2$ *then*

$$[\operatorname{im}\psi_x : L\langle j\rangle] = 0$$

and otherwise

$$[\operatorname{im}\psi_x : L\langle j\rangle] \geqslant \min\left\{[\mathcal{P}_x : L\langle j\rangle], [\mathcal{P}_{sx} : L\langle j\rangle] - \sum_{k}^{l(w_o x)} ([\mathcal{P}_x : L\langle j+2k\rangle])\right\}.$$

(iv) *Let* $C(x, L)$ *be the multiset of composition factors of* $\mathcal{P}_x$ *isomorphic to some* $L\langle j\rangle$, $j \in \mathbb{Z}$. *There is a bijection* $c : C(w_o, L) \longrightarrow C(x, L)$ *such that*

$$[\operatorname{im}\psi_x : L\langle j\rangle] = |\{L\langle j\rangle \mid c(L\langle j\rangle) = L\langle j\rangle\}|.$$

Proof. (i) The composition of maps is obviously non-zero, since its image always contains the simple composition factor corresponding to the dominant highest weight. On the other hand, the homomorphism space in question is one-dimensional.

(ii) The first statement follows directly from the definition of a graded homomorphism. For the second statement we refer to the next section (Theorem 6.1). We prove the last formula by induction on k. Let $f = f_{sx,x}$ and $m = m_{L,sx}$. If $k = 0$, we get by the four-term exact sequence and the definition of $m_{L,sx}$

$$\begin{aligned}[\operatorname{im} f : L\langle m\rangle] &= [\mathcal{P}_x : L\langle m\rangle] - [\operatorname{coker} f : L\langle m\rangle]\\ &= [\mathcal{P}_x : L\langle m\rangle] - [\ker f : L\langle m+2\rangle]\\ &= [\mathcal{P}_x : L\langle m\rangle].\end{aligned}$$

This is exactly (4.3) (with the convention for $k = 0$ that the occurring sum is zero). We now assume the formula to be true for k; hence

$$\begin{aligned}&[\operatorname{im} f : L\langle m-2(k+1)\rangle]\\ &\quad= [P_x : L\langle m-2(k+1)\rangle] - [\operatorname{coker} f : L\langle m-2(k+1)\rangle]\\ &\quad= [P_x : L\langle m-2(k+1)\rangle] - [\ker f : L\langle m-2k\rangle]\\ &\quad= [P_x : L\langle m-2(k+1)\rangle] - [\mathcal{P}_{sx} : L\langle m-2k\rangle] + [\operatorname{im} f : L\langle m-2k\rangle]\\ &\quad= [P_x : L\langle m-2(k+1)\rangle] - [\mathcal{P}_{sx} : L\langle m-2k\rangle] + [P_x : L\langle m-2k\rangle]\\ &\qquad+ \sum_{j=0}^{k-1}\big([\mathcal{P}_x : L\langle m-2j\rangle] - [\mathcal{P}_{sx} : L\langle m-2j\rangle]\big)\\ &\quad= [P_x : L\langle m-2(k+1)\rangle] + \sum_{j=0}^{k}\big([\mathcal{P}_x : L\langle m-2j\rangle] - [\mathcal{P}_{sx} : L\langle m-2j\rangle]\big),\end{aligned}$$

which is the required formula.

(iv) Because of the four-term exact sequence (4.2), the morphism $f_{sx,x}$ provides a bijection $c_{sx,x} : C(sx, L) \longrightarrow C(x, L)$ such that $c_{sx,x}(L\langle j\rangle) \in \{L\langle j\rangle, L\langle j-2\rangle\}$ and $[\operatorname{im} f_{x,sx} : L\langle j\rangle] = |\{L\langle j\rangle \mid c_{sx,x}(L\langle j\rangle) = L\langle j\rangle\}|$. The claim follows then inductively using the composition from (i).

(iii) The first part is true, since ψ_x is homogeneous of degree zero. For the parity condition we refer to Theorem 6.1. The second part follows directly from the construction of the bijection c from (iv). □

Corollary 4.5. *Let* $y \in W$ *and suppose that* $[\Delta(0) : L(y \cdot 0)] = 1$. *Then we have equality in Theorem* 4.1 *for any* $x \in W$.

Proof. This follows directly from equation (4.3). □

Note that the bijection in Theorem 4.4(iv) distinguishes somehow the composition factors occurring with higher multiplicities, so that for a Duflo involution x the composition factors of $\mathcal{U}/\operatorname{Ann}_{\mathcal{U}} L(x) \bigotimes_{\mathcal{U}} \Delta(0)$ are just those which appear in the corresponding principal series representation $\mathcal{P}_x$ and in $\mathcal{P}_{w_o}$ in the same degree.

5. *Principal series modules: some explicit examples*

For Lie algebras of rank 2 it is possible to describe explicitly the principal series modules via representations of a quiver. The quivers with relations can be found in [**24**]. An explanation on how to compute these representations is included in [**23**]. A module is called *rigid* if its socle and radical filtrations coincide (see [**14**]). It is a non-obvious fact that Verma modules are rigid (see [**3–5**, **13**]). We now give examples where the principal series modules are *not* rigid. We list the socle, radical and grading filtrations denoted by S, R and G respectively. (Note that at this point we use the fact that the algebra A is positively graded so that the grading filtration is well-defined.)

For type A_2, all the $\mathcal{P}_x$ are rigid. In type G_2 we get a similar picture as for B_2. For higher ranks there should also be some principal series representation for which no two of these filtrations coincide.

5.1. *The case* B_2

We denote the two simple reflections by s and t. (For our purpose it is not important which reflection corresponds to the long root.) We list the composition factors occurring in each layer of the filtration. The layers are separated by the symbol '$<$' and L_x denotes the simple module corresponding to $L(x \cdot 0)$. By the duality Theorem 3.4 it is sufficient to consider the following four modules.

$$\begin{array}{rcll}
\mathcal{P}_e \cong d\mathcal{P}_{stst} & : & L_e < L_s, L_t < L_{st}, L_{ts} < L_{sts}, L_{tst} < L_{stst} & \text{(SRG)} \\
\mathcal{P}_s \cong d\mathcal{P}_{tst} & : & L_s < L_e, L_{ts}, L_{st} < L_t, L_{tst}, L_{sts} < L_{stst} & \text{(SG)} \\
 & & L_s < L_{ts}, L_{st} < L_e, L_{tst}, L_{sts} < L_t, L_{stst} & \text{(R)} \\
\mathcal{P}_t \cong d\mathcal{P}_{sts} & : & L_t < L_e, L_{ts}, L_{st} < L_s, L_{tst}, L_{sts} < L_{stst} & \text{(SG)} \\
 & & L_t < L_{ts}, L_{st} < L_e, L_{tst}, L_{sts} < L_s, L_{stst} & \text{(R)} \\
\mathcal{P}_{st} \cong d\mathcal{P}_{ts} & : & L_t, L_{tst} < L_e, L_{ts}, L_{st}, L_{stst} < L_s, L_{sts}. & \text{(SRG).}
\end{array}$$

6. *Rigidity, parity property and Koszul modules*

Let us have a closer look at the three filtrations. A general necessary and sufficient condition for coincidence of all three filtrations is given in [**5**, Proposition 2.4.1] (see also [**23**, Lemma 5.3.1]). The geometric approach in [**10**] and [**9**, (3.1)] gives the same filtration on principal series modules as the one associated to our grading. This was the motivation for us to prove the following parity property of the grading filtration using the graded four-step exact sequence (4.2) and the fact that the lift $\mathcal{P}_{w_o}$ of the projective Verma module has this property.

THEOREM 6.1 (parity property). *Let $x \in W$. The (graded) composition factors of $\mathcal{P}_x$ satisfy the condition*

$$[\mathcal{P}_x : \tilde{L}(y \cdot 0)\langle i\rangle] \neq 0 \Longrightarrow l(y) - i \equiv 0 \bmod 2.$$

In particular, in each layer of the grading filtration, there appear only simple modules corresponding to elements of the Weyl group whose lengths are all of the same parity.

Proof. The graded version of the Duflo–Zhelobenko sequence (4.2) gives the proof inductively, since the result is well known for the projective Verma module (see for example [**5**]). □

Let $N \in \operatorname{gmof} - A$ (or more generally a graded module over some positively graded ring A where A_0 is semisimple) be called a *Koszul module* if

$$\operatorname{Ext}^i_{\operatorname{gmof}-A}(M, L) = 0$$

holds for all simple modules L not concentrated in degree i. In [**5**] it is shown that the standard lifts of all Verma modules are Koszul modules. This is not the case for principal series modules in general. A counter-example is given in type B_2: the head of $\mathcal{P}_s$ is not concentrated in one single degree; in particular it is not concentrated in degree zero.

Although principal series modules do not satisfy the assumption of the following theorem, the theorem seems nevertheless to be quite natural and provides also an application of the adjointness theorem of graded translation functors. For a semi-regular weight $\lambda \in W$ we denote by $\tilde{\theta}_0^\lambda$ the graded version of translation onto the wall (see [**26**, Theorem 8.1]).

THEOREM 6.2. *Let $M \in \operatorname{gmof} - A$ be Koszul and assume that $\tilde{\theta}_0^\lambda M$ is also Koszul with $W_\lambda = \{1, s\}$. Moreover, we assume the existence of a short exact sequence of graded modules of the form*

$$M\langle 1\rangle \hookrightarrow \tilde{\theta}_s M \twoheadrightarrow N. \tag{6.1}$$

Then the quotient N is also a Koszul module.

Proof. Recall the adjointnesses

$$\left(\tilde{\theta}_0^\lambda, \tilde{\theta}_\lambda^0\langle -1\rangle\right) \quad \text{and} \quad \left(\tilde{\theta}_\lambda^0, \tilde{\theta}_0^\lambda\langle 1\rangle\right)$$

of [**26**, Theorem 8.4]. The exact sequence (6.1) implies an exact sequence in $\operatorname{gmof} - A$ of the form

$$\operatorname{Ext}^i(M\langle 1\rangle, L\langle n\rangle) \longleftarrow \operatorname{Ext}^i(\tilde{\theta}_s M, L\langle n\rangle) \longleftarrow \operatorname{Ext}^i(N, L\langle n\rangle) \longleftarrow \operatorname{Ext}^{i-1}(M\langle 1\rangle, L\langle n\rangle) \tag{6.2}$$

for some simple module $L\langle n\rangle$ concentrated in degree n. The second term can be reformulated using the adjointness property as

$$\operatorname{Ext}^i(\tilde{\theta}_s M, L\langle n\rangle) \cong \operatorname{Ext}^i\left(\tilde{\theta}_0^\lambda M, \tilde{\theta}_0^\lambda L\langle n+1\rangle\right). \tag{6.3}$$

Either $\tilde{\theta}_0^\lambda L = 0$, and therefore the right-hand side of (6.3) is zero or $\tilde{\theta}_0^\lambda L$ is a simple module (see [**16**, 4.12, (3)]). In the latter case (6.3) is equal to $\operatorname{Ext}^i(\tilde{\theta}_0^\lambda M, \tilde{L}\langle n\rangle)$ for some simple module $\tilde{L}$ of degree zero (see [**26**]). Provided furthermore that $n \neq i$, the extension (6.3) is trivial by the assumptions of the theorem.

Hence, with the assumptions on M and for $n \neq i$, the sequence (6.2) yields an exact sequence of the form

$$0 \longleftarrow \mathrm{Ext}^i(N, L\langle n\rangle) \longleftarrow 0. \tag{6.4}$$

Therefore, all the extensions of a simple module L of degree i by the module N have to be of degree i. That is, N is a Koszul module. □

COROLLARY 6.3. *Let $x \in W$ and let s be a simple reflection. The exact sequence*

$$\mathcal{P}_{x,w_o}\langle 1\rangle \hookrightarrow \tilde{\theta}_s \mathcal{P}_{x,w_o} \twoheadrightarrow \mathcal{P}_{x,w_o s}$$

of Theorem 2.2 *implies that $\mathcal{P}_{x,w_o s}$ is a Koszul module.*

Proof. In this case the module on the left is the (standard) lift of a Verma module, and therefore Koszul. Translation onto the wall gives a lift of a Verma module with simple head in degree zero (see [**26**]). Therefore in this case the assumptions of the theorem are fulfilled and the statement follows. □

REMARK 6.4. In [**1**], the authors defined a filtration on principal series modules. We conjecture that the filtration associated to our grading induces exactly the filtration of [**1**]. This is well known for Verma modules [**4**] and hence for dual Verma modules. In general, the result might follow using techniques from [**10**].

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References

1. H. ANDERSEN and N. LAURITZEN, 'Twisted Verma modules', *Studies in memory of Issai Schur*, Progress in Mathematics 210 (Birkhäuser, 2002) 1–26.

2. H. ANDERSEN, J.-C. JANTZEN and W. SOERGEL, 'Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p', *Astérisque* 220 (Societé Mathématique de France, Paris, 1994).

3. D. BARBASCH, 'Filtrations on Verma modules', *Ann. Sci. École Norm. Sup.* (4) 16 (1983) 489–494.

4. A. BEILINSON and J. BERNSTEIN, 'A proof of Jantzen conjectures', *I. M. Gelfand Seminar*, Advances in Soviet Mathematics 16 – Part 1 (American Mathematical Society, Providence, RI, 1993) 1–50.

5. A. BEILINSON, V. GINZBURG and W. SOERGEL, 'Koszul duality patterns in representation theory', *J. Amer. Math. Soc.* 9 (1996) 473–527.

6. I. BERNSTEIN and S. I. GELFAND, 'Tensor products of finite and infinite dimensional representations of semisimple Lie algebras', *Compositio Math.* 41 (1980) 245–285.

7. I. N. BERNSTEIN, I. M. GELFAND and S. I. GELFAND, 'A category of $\mathfrak{g}$-modules', *Funct. Anal. Appl.* 10 (1976) 87–92.

8. N. BOURBAKI, Groupes et algèbres de Lie (Masson, 1994).

9. L. CASIAN and D. COLLINGWOOD, 'Complex geometry and the asymptotics of Harish–Chandra modules of real reductive Lie groups. III: Estimates on $\mathfrak{n}$-homology', *J. Algebra* 116 (1988) 415–456.

10. L. CASIAN and D. COLLINGWOOD, 'Weight filtrations for induced representations of real reductive Lie groups', *Adv. Math.* 73 (1989) 79–144.

11. J. Dixmier, *Enveloping algebras*, Graduate Studies in Mathematics 11 (American Mathematical Society, Providence, RI, 1996).

12. M. Duflo, 'Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algebre de Lie semi-simple', *Ann. Math.* 105 (1977) 107–120.

13. R. S. Irving, 'The socle filtration of a Verma module', *Ann. Sci. École Norm. Sup.* 21 (1988) 47–65.

14. R. S. Irving, 'Singular blocks of category $\mathcal{O}$', *Math. Z.* (2) 204 (1990) 209–224.

15. J.-C. Jantzen, *Modul mit einem höchsten Gewicht* (Springer, 1979).

16. J.-C. Jantzen, *Einhüllende Algebren halbeinfacher Liealgebren* (Springer, 1983).

17. A. Joseph, 'The Enright functor on the Bernstein–Gelfand–Gelfand category $\mathcal{O}$', *Invent. Math.* 67 (1982) 423–445.

18. A. Joseph, 'Completion functors in the $\mathcal{O}$ category', *Non-commutative harmonic analysis and Lie groups*, Lecture Notes in Mathematics 1020 (Springer, 1983) 80–106.

19. A. Joseph, 'Problems old and new', *In honor of Bertram Kostant* (ed. J.-L. Brylinski, Birkhäuser, 1994) 385–414.

20. D. Kazhdan and G. Lusztig, 'Representations of Coxeter groups and Hecke algebras', *Invent. Math.* 53 (1979) 165–184.

21. G. Lusztig, 'Characters of reductive groups over a finite field', Annals of Mathematics Studies 107 (Princeton University Press, Princeton, NJ, 1984).

22. W. Soergel, 'Kazhdan–Lusztig polynomials and a combinatoric for Tilting modules', *Represent. Theory* 1 (1997) 83–114.

23. C. Stroppel, 'Der Kombinatorikfunktor $\mathbb{V}$: Graduierte Kategorie $\mathcal{O}$, Hauptserien und primitive Ideale', Dissertation, Universität Freiburg i. Br., 2001.

24. C. Stroppel, 'Category $\mathcal{O}$: quivers and endomorphism rings of projectives', *Represent. Theory* 7 (2003) 322–345.

25. C. Stroppel, 'Homomorphisms and extensions of principal series', *J. Lie Theory* 13 (2003) 193–212.

26. C. Stroppel, 'Category $\mathcal{O}$: gradings and translation functors', *J. Algebra* 268 (2003) 301–326.

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