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Preprojektive algebraer og n-Calabi-Yau fullføringer

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SAMMENDRAG

Vi viser at den preprojektive algebraen $\mathcal{P}_k(Q)$ til et kogger Q er isomorf med tensor algebraen $T_{kQ}(\theta)$, hvor $\theta = \mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), kQ)$. I tillegg konstruerer vi en kvasi-isomorfi mellom $\Pi_n(A)$ og $V_n(\Pi_n(A))$, hvor $\Pi_n(A)$ er n-Calabi-Yau fullføringen av en homologisk glatt algebra A og V_n er det n'te skiftet av den induserte predualitets funktoren på kategorien av bimoduler over $\Pi_n(A)$. Til slutt viser vi at 2-Calabi-Yau fullføringen $\Pi_2(kQ)$ av veialgebraen kQ er kvasi-isomorf til den preprojektive algebraen $\mathcal{P}_k(Q)$.

ABSTRACT

We show that the preprojective algebra $\mathcal{P}_k(Q)$ of a quiver Q is isomorphic to the tensor algebra $T_{kQ}(\theta)$, where $\theta = \mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), kQ)$. We also construct a quasi-isomorphism between $\Pi_n(A)$ and $V_n((\Pi_n(A))$, where $\Pi_n(A)$ is the n-Calabi-Yau completion of a homologically smooth algebra A and V_n is the nth shift of the induced preduality functor on the category of bimodules over $\Pi_n(A)$. Finally we show that the 2-Calabi-Yau completion $\Pi_2(kQ)$ of the path algebra kQ is quasi-isomorphic to the preprojective algebra $\mathcal{P}_k(Q)$.

PREFACE

This thesis was written under the supervision of Prof. Aslak Bakke Buan at the Department of Mathematical Sciences, NTNU. It marks the end of my time as a master student in mathematics at NTNU

Firstly I want to express my deepest gratitude towards my advisor Aslak Bakke Buan for providing me with such an interesting task and for doing his utmost in order to answer my questions. I also want to thank postdoc Yu Qiu for fruitful discussions and for helpful feedback on drafts of my thesis. Finally I want to thank my parents and my friends for their encouragement throughout these years.

CONTENTS

1. INTRODUCTION

1.1 Purpose and motivation

There are two main objects of study in this thesis, the preprojective algebra of a quiver, and the n-Calabi-Yau completion of a homologically smooth algebra. The preprojective algebra of a quiver Q is given by

$$\mathcal{P}_k(Q) = k\bar{Q}/(\rho)$$

where $\bar{Q}$ is the quiver obtained by adding an arrow $\alpha^* : y \to x$ for each arrow $\alpha : x \to y$ in Q, and where

$$\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*] = \sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha)$$

The preprojective algebra appears in diverse situation, for example by Kronheimer in [12] for studying problems in differential geometry, and in Lusztig's perverse sheaf approach to quantum groups (see [13], [14], [15]). There is a well known isomorphism

$$\mathcal{P}_k(Q) \cong T_{kQ}(\theta)$$

where $T_{kQ}(\theta)$ is the tensor algebra of the module $\theta = \mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), kQ)$. We will give a more detailed version of the proof in [3] for this. It is also well known that $\mathcal{P}_k(Q)$ is the sum of the preprojective modules of kQ. We will give a proof of this in subsection 2.4 using the isomorphism above. This will in particular imply that $\mathcal{P}_k(Q)$ is finite dimensional if and only if Q is Dynkin.
Let A be a finite dimensional algebra over the field k. We say that A is *homologically smooth* if it has finite global dimension. In the second part of the thesis we will consider the *n-Calabi-Yau completion* $\Pi_n(A)$ of a finite dimensional homologically smooth algebra A. This is the differential graded algebra

$$\Pi_n(A) = T_A(\theta_A) = A \oplus \theta_A \oplus \theta_A \otimes_A \theta_A \oplus ...$$

where $\theta_A = \Theta_A[n-1]$ and Θ_A is the *inverse dualizing complex* of A given by taking a homotopically projective resolution of the chain complex

$$\mathrm{RHom}_{A^e}(A, A^e)$$

where $A^e = A \otimes_k A^{\mathrm{op}}$. This is a special case of what Keller studies in section 4 of [7]. Let $A = \mathrm{End}_{\mathcal{C}}(T)$ where $\mathcal{C}$ is the derived category of quasi-coherent sheaves on a smooth algebraic variety X of dimension $n-1$ and T is a tilting object in $\mathcal{C}$. Then the derived category of quasi-coherent sheaves on the total space of the canonical bundle of X is triangle equivalent to the derived category of $\Pi_n(A)$ (see [19]). Also the n-Calabi-Yau completion corresponds to Ed Segal's cyclic composition under Koszul diality (see [18]). Furthermore in [6] Keller claims that the preprojective algebra $\mathcal{P}_k(Q)$ of a non-Dynkin quiver Q is quasi-isomorphic to $\Pi_2(kQ)$, the 2-Calabi-Yau completion of the path algebra kQ, which gives an equivalence of the derived categories $D(\Pi_2(kQ))$ and $D(\mathcal{P}_k(Q))$. We will show this in subsection 5.3.

Calabi-Yau categories (see [6] for the definition) plays an important in homological mirror symmetry and in categorification of cluster algebras. The category of modules over the preprojective algebra has a 2-Calabi-Yau property which Geiss-Leclerc-Schrer uses (see [4]). Also Kontsevich-Soibelman interpretation of cluster transformations for studying Donaldson-Thomas invariants and stability structures uses the Calabi-Yau property (see [11]). In subsection 4.8 of [7] Keller proves that $D^b(\Pi_n(A))$ is n-Calabi-Yau by showing the existence of a quasi-isomorphism

$$f : B \to \mathrm{RHom}_{B^e}(B, B^e)[n]$$

where $B = \Pi_n(A)$. We will give a proof of this result based on Keller's proof, but with more details. Unfortunately, due to time limits there is one detail of the proof which we won't verify is true.

1.2 Contents

The main object of study in section 2 is $\mathcal{P}_k(Q)$, the preprojective algebra of a quiver. We show the well known result that $\mathcal{P}_k(Q)$ is isomorphic to $T_{kQ}(\theta)$ (Theorem 2.13) where $\theta = \mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), kQ)$. Our proof is a more detailed version of the one given in [3][1]. In order to do this we need introduce the concept of derivations. As a necessary tool we also show that any module over the path algebra kQ has a canonical projective resolution. Finally in subsection 2.4 we use Theorem 2.13 to show that the preprojective algebra is finite dimensional if and only if Q is Dynkin.

In section 3 we introduce differential graded algebras and modules. We define the shift $M[n]$ of a dg module, the tensor product $M \otimes_A N$ of two dg modules, and the chain complex $\mathcal{H}\mathrm{om}_A(M, N)$ of graded maps between M and N. We also investigate how these operations interact. We define differential graded categories in 3.6, and we show that we have a dg category $\mathcal{C}_{dg}(A)$ with morphism sets $\mathcal{H}\mathrm{om}_A(M, N)$ in 3.7. We describe the homotopy category and the derived category of a dg algebra in 3.9 and 3.10. All of this material is well known and can also be found in [8], [9] and [10].

[1] For a different proof see [17]

In section 4 we investigate preduality functors. We show in proposition 4.2 that a dg algebra A equipped with an involution $\tau : A \to A^{\mathrm{op}}$ gives us a preduality functor on $\mathcal{C}_{dg}(A)$. We also prove that the shift of a preduality functor is still a preduality functor (proposition 4.3). Finally in 4.4 we investigate how the preduality functors relate when we have a morphism of dg algebras which commute with the involution. Most of this material is taken from [7]

We define the n-Calabi-Yau completion B of a homologically smooth algebra A in section 5. Our main theorem is 5.1, which has as a consequence that the derived category of B is n-Calabi-Yau. Except for one detail we give a complete proof of Theorem 5.1 in subsection 5.2. It is based on the proof given for Theorem 4.8 in [7]. In subsection 5.3 we show that the preprojective algebra of a non-Dynkin quiver is quasi-isomorphic to the 2-Calabi-Yau completion of the the path algebra.

1.3 Notation and terminology

Throughout this thesis k will always be a commutative ring. In section 2, 4 and 5 k will be a field.

A k-algebra is a ring A together with a ring morphism

$$f : k \to A$$

such that $\operatorname{Im} f \subset Z(A)$ where $Z(A)$ is the center of A. We let $\operatorname{Mod} A$ ($\operatorname{mod} A$) denote the category of (finitely generated) modules over the k-algebra A. When k is a field we have the duality functor $D = \operatorname{Hom}_k(-, k) : \operatorname{mod} A \to \operatorname{mod}(A^{\mathrm{op}})$.

A *graded k-algebra* A is a graded k-module

$$A = \bigoplus_{i \in \mathbb{Z}} A^i$$

with an algebra structure such that if $a \in A^i$ and $b \in A^j$, then $a \cdot b \in A^{i+j}$.

An *A-algebra* is a ring B together with a ring morphism $f : A \to B$. If B_1 and B_2 are A-algebras with morphisms $f_1 : A \to B_1$ and $f_2 : A \to B_2$ then a A-algebra morphism $\phi : B_1 \to B_2$ is a ring morphism satisfying $\phi \circ f_1 = f_2$.

If M is an A-bimodule satisfying $r \cdot m = m \cdot r$ for all $r \in k$, $m \in M$ we have the *tensor algebra*

$$T_A(M) = \bigoplus_{n=0}^{\infty} M^n$$

where $M^n = M \otimes_A M \otimes_A ... \otimes_A M$ is the tensor product taken n times. This is a graded k-algebra with multiplication given by

$$m \cdot n = m \otimes n \in M^m \otimes_A M^n = M^{m+n}$$

for $m \in M^m$ and $n \in M^n$. Since $T_A(M)$ contains A as a subalgebra it will be an A-algebra. The ideal

$$\bigoplus_{n=1}^{\infty} M^n$$

of $T_A(M)$ is called the *augmentation ideal*. If B is an A-algebra and $f : M \to B$ is a morphism of A-bimodules we get an induced morphism $F : T_A(M) \to B$ of A-algebras given by

$$F(m_1 \otimes m_2 \otimes ... \otimes m_n) = f(m_1) \cdot f(m_2) \cdot ... \cdot f(m_n)$$

A *chain complex* M over a ring A is a graded A module

$$M = \bigoplus_{i=-\infty}^{\infty} M^i$$

with a differential $d : M \to M$ of degree 1 satisfying $d \circ d = 0$. We denote by

$$d^n : M^n \to M^{n+1}$$

the restriction of d to M^n. We set $Z^n(M) = \operatorname{Ker} d^n$ and $B^{n+1}(M) = \operatorname{Im} d^n$. Since $d^2 = 0$ we have that $B^n(M) \subset Z^n(M)$. The nth *homology* of M is

$$H^n(M) = Z^n(M)/B^n(M)$$

A chain map $f : M \to N$ is a k module morphism satisfying $f \circ d = d \circ f$. A *quasi-isomorphism* is a chain map that induces an isomorphism in homology.

If we have two chain complexes M and N we can form the chain comples $M \otimes_k N$ with

$$(M \otimes_k N)^n = \bigoplus_{i+j=n} M^i \otimes_k N^j$$

and where the differential is given by

$$d(m \otimes n) = dm \otimes n + (-1)^{|m|} \cdot m \otimes dn$$

There is a natural isomorphism of chain complexes

$$\tau : M \otimes_k N \to N \otimes_k M$$

defined by

$$\tau(m \otimes n) = (-1)^{|m| \cdot |n|} \cdot (n \otimes m) \tag{1.1}$$

2. THE PREPROJECTIVE ALGEBRA OF A QUIVER

In this section all modules will be left modules unless stated otherwise.

Let $Q = (Q_0, Q_1)$ be a finite quiver without cycles, where Q_0 are the set of vertices and Q_1 are the set of arrows. Let k be a field. We have a quiver $\bar{Q}$ obtained from Q by adding an arrow $\alpha^* : j \to i$ for each arrow $\alpha : i \to j$ in Q. The *preprojective algebra* of the quiver Q is defined as

$$\mathcal{P}_k(Q) = k\bar{Q}/(\rho)$$

where

$$\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*] = \sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha)$$

(see [17]). Note that kQ can be identified as a subalgebra of $\mathcal{P}_k(Q)$. Consider the kQ-bimodule $\mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), kQ)$ It has a natural kQ bimodule structure, where the left module structure is inherited from the right kQ module structure of $D(kQ_{kQ})$, and the right structure is inherited from the right module structure of kQ. The main goal in this section is to show that there exists an isomorphism

$$\mathcal{P}_k(Q) \cong T_{kQ}(\theta)$$

which acts as identity on kQ, and which maps the arrows in Q_1^* onto the augmentation ideal of $T_{kQ}(\theta)$, where $\theta = \mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), kQ)$ The subsections 2.1, 2.2 and 2.3 will be devoted to developing the necessary tools, and proving this result. It is an expanded version of the proof of Theorem 3.1 in [3]. In subsection 2.4 we will show that the preprojective algebra is finite dimensional if and only if Q is a Dynkin quiver.

2.1 Derivations

Here A will be a k-algebra, where k is a field. The A bimodule $A^e = A \otimes_k A^{\mathrm{op}}$ will have a natural ring structure given by

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$$

for $a, c \in A$, $b, d \in A^{\mathrm{op}}$. A bi-A-module M will be left A^e-module via

$$(a \otimes b) \cdot m = a \cdot m \cdot b$$

Conversly any left A^e-module will also be a bi-A-module in the natural way.

Definition 2.1. Let M be an A bimodule. A **k-derivation**

$$\mathcal{D} : A \to M$$

is a k-linear map such that

$$\mathcal{D}(a \cdot b) = \mathcal{D}(a) \cdot b + a \cdot \mathcal{D}(b)$$

The collection of all k-derivations from A to M will be denoted by $\mathrm{Der}(A, M)$.

We want to construct a k-derivation satisfying a universal property. Consider the A-bimodule

$$A \otimes_k A$$

with module structure given by

$$a_1(a \otimes b)a_2 = a_1 a \otimes b a_2$$

for $a, a_1, a_2, b \in A$. The multiplication map

$$m : A \otimes_k A \to A \quad \text{with} \quad m(a \otimes b) = a \cdot b$$

will then be an A-bimodule morphism. Therefore the kernel

$$\Omega^1 A = \ker(m)$$

will be an A bimodule. There is also a natural k derivation

$$d : A \to \Omega^1 A$$

given by

$$d(a) = a \otimes 1 - 1 \otimes a$$

Proposition 2.2. *Let $\mathcal{D} : A \to M$ be a k-derivation. Then there exists a unique A^e morphism*

$$\Theta_{\mathcal{D}} : \Omega^1 A \to M$$

such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathcal{D}"] \arrow[d, "d"'] & M \\
\Omega^1 A \arrow[ur, "\Theta_{\mathcal{D}}"']
\end{tikzcd}$$

commutes.

This results implies that there is a natural isomorphism

$$\mathrm{Hom}_{A^e}(\Omega^1 A, M) \cong \mathrm{Der}(A, M)$$

given by sending $f \in \mathrm{Hom}_{A^e}(\Omega^1 A, M)$ to $f \circ d \in \mathrm{Der}(A, M)$.

Proof. Assume $\mathcal{D}$ is a derivation from A to M. This induces a left A-module morphism

$$\bar{\mathcal{D}} : A \otimes_k A \to M$$

given by

$$\bar{\mathcal{D}}(a \otimes b) = a \cdot \mathcal{D}(b)$$

for $a, b \in A$. Composing this with the inclusion of $\Omega^1 A$ into $A \otimes_k A$ we get a left A morphism

$$\Theta_{\mathcal{D}} : \Omega^1 A \to M$$

We show that this will also be a right A module morphism. So assume

$$\sum_{i=1}^{n} a_i \otimes b_i \in \Omega^1 A \quad \text{so} \quad \sum_{i=1}^{n} a_i \cdot b_i = 0$$

and let $b \in A$. The right action is given by

$$\sum_{i=1}^{n} (a_i \otimes b_i) \cdot b = \sum_{i=1}^{n} a_i \otimes (b_i \cdot b)$$

Now we get that

$$\Theta_{\mathcal{D}}(\sum_{i=1}^{n} a_i \otimes (b_i \cdot b)) = \sum_{i=1}^{n} a_i \cdot \mathcal{D}(b_i \cdot b) = \sum_{i=1}^{n} a_i \cdot \mathcal{D}(b_i) \cdot b + \sum_{i=1}^{n} a_i \cdot b_i \cdot \mathcal{D}(b)$$

and since

$$\sum_{i=1}^{n} a_i \cdot b_i \cdot \mathcal{D}(b) = 0 \quad \text{and} \quad \sum_{i=1}^{n} a_i \cdot \mathcal{D}(b_i) \cdot b = \Theta_{\mathcal{D}}(\sum_{i=1}^{n} a_i \otimes b_i) \cdot b$$

it follows that

$$\Theta_{\mathcal{D}}(\sum_{i=1}^{n} a_i \otimes (b_i \cdot b)) = \Theta_{\mathcal{D}}((\sum_{i=1}^{n} a_i \otimes b_i)) \cdot b$$

so $\Theta_{\mathcal{D}}$ is a right A module morphism. Now a simple calculation also gives us that

$$\Theta_{\mathcal{D}} \circ d = \mathcal{D}$$

and hence we have shown the existence part of the theorem. Uniqueness comes from the fact that the image of d generates $\Omega^1 A$ as an A-bimodule. $\square$

Now let $M = A \otimes_k A$. Then $\mathrm{Der}(A, A \otimes_k A)$ has a A-bimodule structure. Let $\Delta \in \mathrm{Der}(A, A \otimes_k A)$ be the derivation given by

$$\Delta(a) = a \otimes 1 - 1 \otimes a$$

The sub A-bimodule of $\mathrm{Der}(A, A \otimes_k A)$ generated by Δ will be denoted by $A\Delta A$. We want to show that there is an A-bimodule isomorphism

$$\mathrm{Der}(A, A \otimes_k A)/A\Delta A \cong \mathrm{Ext}^1_A(D(A), A)$$

In order to do this we need some preliminary results.

Lemma 2.3. *Let X be a left A module and Y a finite dimensional right A module. Then there exists A-bimodule isomorphisms*

$$X \otimes_k Y \xrightarrow{\eta} \mathrm{Hom}_A(A \otimes_k DY, X)$$

given by

$$\eta(x \otimes y)(a \otimes f) = f(y)a \cdot x$$

with $x \in X$, $y \in Y$, $a \in A$ and $f \in DY$, and where the left A module structure of $A \otimes_k DY$ comes from the left A module structure of A, the left A module structure of $\mathrm{Hom}_A(A \otimes_k DY, X)$ comes from the right A module structure of A, and the right A module structure of $\mathrm{Hom}_A(A \otimes_k DY, X)$ comes from the left A module structure of DY.

Proof. It is easy to see that η is an A bimodule morphism, so we only need to show that η is bijective. We note first that Y and DY have the same dimension over k, which we will call n. We choose a basis e_1, e_2, ... e_n for Y, which gives us a dual basis e_1^*, e_2^*, ... e_n^* for DY. This induces an isomorphism

$$A \otimes_k DY \cong \bigoplus_{i=1}^n A$$

given by sending $a \otimes e_i^*$ to $(0, 0, ..., 0, a, 0, ..., 0)$ where a is in component i of $\bigoplus_{i=1}^n A$. As k-vector spaces we therefore have isomorphisms

$$\mathrm{Hom}_A(A \otimes_k DY, X) \cong \mathrm{Hom}_A(\bigoplus_{i=1}^n A, X) \cong \bigoplus_{i=1}^n \mathrm{Hom}_A(A, X) \cong \bigoplus_{i=1}^n X$$

We also have a k-isomorphism

$$\bigoplus_{i=1}^n X \cong X \otimes_k Y$$

given by sending $(x_1, x_2, ..., x_n)$ to $x_1 \otimes e_1 + x_2 \otimes e_2 + ... + x_n \otimes e_n$. We denote the composition of these maps by δ, so

$$\delta : \mathrm{Hom}_A(A \otimes_k DY, X) \to X \otimes_k Y$$

Now a simple calculation shows that $\eta \circ \delta = 1$ and $\delta \circ \eta = 1$, so η is bijective. Since η is also an A bimodule morphism we get the result. □

Lemma 2.4. *Let X be a left A module and Y a finite dimensional right A module. There is a natural isomorphism*

$$\sigma : X \otimes_k Y \to \mathrm{Hom}_k(DY, X)$$

of A bimodules, given by

$$\sigma(x \otimes y)(f) = f(y) \cdot x$$

Proof. As before we choose a basis for Y, giving a dual basis for DY. Then the inverse for σ is given by the composition of the induced k isomorphisms

$$\mathrm{Hom}_k(DY, X) \cong \bigoplus_{i=1}^{n} X$$

and

$$\bigoplus_{i=1}^{n} X \cong X \otimes_k Y$$

so σ is bijective and the result follows. □

We will let $\mathrm{Mod}\, A$ denote the category of right A-modules and $\mathrm{Mod}\,(A^{\mathrm{op}})$ denote the category of left A-modules (not necessarily finite dimensional).

Lemma 2.5. *Let X be a left A module and Y a finite dimensional right A module. There is a natural isomorphism*

$$\mathrm{Hom}_A(\Omega^1 A \otimes_A DY, X) \cong \mathrm{Hom}_{A^e}(\Omega^1 A, X \otimes_k Y)$$

Proof. Consider the tensor functor

$$- \otimes_A DY : \mathrm{Mod} A^e \to \mathrm{Mod} A$$

This functor will be left adjoint to the Hom functor

$$\mathrm{Hom}_k(DY, -) : \mathrm{Mod} A \to \mathrm{Mod} A^e$$

Hence there exists a natural isomorphism

$$\mathrm{Hom}_A(\Omega^1 A \otimes_A DY, X) \cong \mathrm{Hom}_{A^e}(\Omega^1 A, \mathrm{Hom}_k(DY, X))$$

Now from lemma 2.4 we get an isomorphism

$$\mathrm{Hom}_{A^e}(\Omega^1 A, \mathrm{Hom}_k(DY, X)) \cong \mathrm{Hom}_{A^e}(\Omega^1 A, X \otimes_k Y)$$

and the result follows. □

Lemma 2.6. *Let X be a left A module and Y a finite dimensional right A module. There is a natural isomorphism*

$$\mathrm{Hom}_A(\Omega^1 A \otimes_A DY, X) \cong \mathrm{Der}(A, X \otimes_k Y)$$

Proof. This follows from proposition 2.2 and lemma 2.6 □

Finally we get the promised result

Proposition 2.7. *There is an isomorphism of A bimodules*

$$\mathrm{Der}(A, A \otimes_k A)/A\Delta A \cong \mathrm{Ext}^1_A(D(A), A)$$

Proof. Consider the exact sequence

$$0 \to \Omega^1 A \xrightarrow{i} A \otimes_k A \xrightarrow{m} A \to 0$$

where i is the inclusion and m is the multiplication map. Since A is a projective A module we have that

$$\mathrm{Tor}^1_A(A, DA) = 0$$

So if we tensor the exact sequence above on the right with DA, we get an exact sequence

$$0 \to \Omega^1 A \otimes_A DA \xrightarrow{i\otimes 1} (A \otimes_k A) \otimes_A DA \xrightarrow{m\otimes 1} A \otimes_A DA \to 0$$

Simplifying, we get

$$0 \to \Omega^1 A \otimes_A DA \to A \otimes_k DA \to DA \to 0$$

Now observe that $A \otimes_k DA$ is isomorphic as a left A module to n copies of A, where n is the dimension of A as a k vector space. In particular $A \otimes_k DA$ will be projective as a left A module. Therefore when we apply the functor $\mathrm{Hom}_A(-, A)$ to the exact sequence above we get the exact sequence

$$0 \to \mathrm{Hom}_A(DA, A) \to \mathrm{Hom}_A(A \otimes_k DA, A) \to \mathrm{Hom}_A(\Omega^1 A \otimes_A DA, A) \to \mathrm{Ext}^1_A(DA, A) \to 0$$

Now we have that

$$\mathrm{Hom}_A(A \otimes_k DA, A) \cong A \otimes_k A \quad \text{and} \quad \mathrm{Hom}_A(\Omega^1 A \otimes_A DA, A) \cong \mathrm{Der}(A, A \otimes_k A)$$

from lemma 2.3 and 2.6. Since the induced map

$$A \otimes_k A \to \mathrm{Der}(A, A \otimes_k A)$$

will take $1 \otimes 1$ to the derivation Δ, the result follows. □

We will also need the following lemma

Lemma 2.8. *Let M be a A-bimodule, and N a sub-bimodule of M. We have a natural isomorphism*

$$T_A(M/N) \cong T_A(M)/(T_A(M) \cdot N \cdot T_A(M))$$

where $T_A(M) \cdot N \cdot T_A(M)$ denotes the $T_A(M)$ bimodule generated by $N \subset T_A(M)$.

Proof. The projection map

$$p : M \to M/N$$

composed with the inclusion map $M/N \subset T_A(M/N)$ induces a surjective A-algebra morphism

$$P : T_A(M) \to T_A(M/N)$$

We want to show that the kernel of P is contained in $T_A(M) \cdot N \cdot T_A(M)$. Since $P(m_1 \otimes m_2 \otimes ... \otimes m_n) = p(m_1) \otimes p(m_2) \otimes ... \otimes p(m_n)$ is a graded map of degree 0 it is enough to show this for elements of $M \otimes_A M \otimes_A ... \otimes_A M$. Observe first that

$$M \otimes_A ... \otimes_A M \otimes_A N \otimes_A M \otimes_A ... \otimes_A M \subset T_A(M) \cdot N \cdot T_A(M)$$

. Now let $x \in M \otimes_A M \otimes_A ... \otimes_A M$. Also let p_i be the natural projection

$$p_i : M \otimes_A M \otimes_A ... \otimes_A M \to M/N \otimes_A ... \otimes_A M/N \otimes_A M \otimes_A ... \otimes_A M$$

where there are $n - i$ terms of M/N and i terms of M. Since $\otimes_A$ is right exact we have the exact sequence

$$\begin{aligned}(M/N \otimes_A ... \otimes_A M/N) \otimes_A N \to (M/N \otimes_A ... \otimes_A M/N) \otimes_A M \\ \to (M/N \otimes_A ... \otimes_A M/N) \otimes_A M/N \to 0\end{aligned}$$

Since $P(x) = 0$ we can find an element $y_1 \in (M/N \otimes_A ... \otimes_A M/N) \otimes_A N$ with image equal to $p_1(x)$. Also since

$$M \otimes_A ... \otimes_A M \otimes_A N \to M/N \otimes_A ... \otimes_A M/N \otimes_A N$$

is surjective we can find

$$x_1 \in M \otimes_A ... \otimes_A M \otimes_A N \subset (M \otimes_A ... \otimes_A M) \cap (T_A(M) \cdot N \cdot T_A(M))$$

such that $p_1(x - x_1) = 0$. Now consider the exact sequence

$$\begin{aligned}(M/N \otimes_A ... \otimes_A M/N) \otimes_A N \otimes_A M \to (M/N \otimes_A ... \otimes_A M/N) \otimes_A M \otimes_A M \\ \to (M/N \otimes_A ... \otimes_A M/N) \otimes_A M/N \otimes_A M \to 0\end{aligned}$$

By a similar argument as before since $p_1(x - x_1) = 0$ we can find element $x_2 \in (M \otimes_A ... \otimes_A M) \cap (T_A(M) \cdot N \cdot T_A(M))$ such that $p_2(x - x_1 - x_2) = 0$.

Repeating this argument we get a sequence of elements $x_1 ... , x_{n-1}$ such that $x_i \in (M \otimes_A ... \otimes_A M) \cap (T_A(M) \cdot N \cdot T_A(M))$ and $p_{n-1}(x - x_1 - ... - x_{n-1}) = 0$. Since we have an exact sequence

$$N \otimes_A M \otimes_A ... \otimes_A M \to M \otimes_A ... \otimes_A M \xrightarrow{p_{n-1}} M/N \otimes_A M \otimes_A ... \otimes_A M \to 0$$

we get that $x - x_1 - ... - x_{n-1} \in T_A(M) \cdot N \cdot T_A(M)$ and therefore $x \in T_A(M) \cdot N \cdot T_A(M)$. It follows that the kernel of P is contained in $T_A(M) \cdot N \cdot T_A(M)$. Since P in an algebra morphism which takes N to 0 we get that $T_A(M) \cdot N \cdot T_A(M)$ is also contained in the kernel of P. Hence P induces an isomorphism

$$P : T_A(M)/(N) \to T_A(M/N)$$

and we are done □

We have the following corollary

Corollary 2.9. *There is a natural isomorphism of rings*

$$T_A(\mathrm{Ext}^1_A(D(A), A)) \cong T_A(\mathrm{Der}(A, A \otimes_k A))/(\Delta)$$

where (Δ) *is the ideal in* $T_A(\mathrm{Der}(A, A \otimes_k A))$ *generated by* Δ

Proof. From proposition 2.7 we have an isomorphism

$$T_A(\mathrm{Ext}^1_A(D(A), A)) \cong T_A(\mathrm{Der}(A, A \otimes_k A)/A\Delta A)$$

Also lemma 2.8 gives us an isomorphism

$$T_A(\mathrm{Der}(A, A \otimes_k A)/A\Delta A) \cong T_A(\mathrm{Der}(A, A \otimes_k A))/(\Delta)$$

so the result follows. □

2.2 Standard projective resolutions

We will now restrict to the ring $A = kQ$, where Q is a finite quiver without cycles. Assume M is a left kQ module. Let i be a vertex of Q, and let e_i denote the idempotent in kQ corresponding to i. Consider the left kQ module $kQe_i \otimes_k e_iM$, where e_iM are all the elements $m \in M$ satisfying $e_i \cdot m = m$, and where kQe_i is left ideal in kQ generated by e_i. Observe that $kQe_i \otimes_k e_iM$ is isomorphic to the sum of n copies of kQe_i, where n is the dimension of e_iM. It is therefore a projective left kQ module. We also have a left kQ-morphism

$$\zeta_i : kQe_i \otimes_k e_iM \to M$$

given by

$$\zeta_i(\rho \otimes m) = \rho \cdot m$$

Taking the sum for all i we get a projective module $\bigoplus_{i\in Q_0} kQe_i \otimes_k e_iM$ and a left kQ-morphism

$$\zeta : \bigoplus_{i\in Q_0} kQe_i \otimes_k e_iM \to M$$

equal to ζ_i on component i. This map is obviously an epimorphism. Now let α be an arrow in Q_1, and let $s\alpha$ and $t\alpha$ denote the scource and target vertex of α. Consider the module $kQe_{t\alpha} \otimes_k e_{s\alpha}M$. As before this will be a projective kQ-module. We have maps

$$\partial_\alpha : kQe_{t\alpha} \otimes_k e_{s\alpha}M \to kQe_{s\alpha} \otimes_k e_{s\alpha}M$$

and

$$\epsilon_\alpha : kQe_{t\alpha} \otimes_k e_{s\alpha}M \to kQe_{t\alpha} \otimes_k e_{t\alpha}M$$

given by

$$\partial_\alpha(\rho \otimes m) = \rho \cdot \alpha \otimes m$$

and

$$\epsilon_\alpha(\rho \otimes m) = \rho \otimes \alpha \cdot m$$

Taking the sum over all α we get a projective module $\bigoplus_{\alpha\in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}M$ and left kQ-morphisms

$$\partial : \bigoplus_{\alpha\in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}M \to \bigoplus_{i\in Q_0} kQe_i \otimes_k e_iM$$

equal to ∂_α on component α, and

$$\epsilon : \bigoplus_{\alpha\in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}M \to \bigoplus_{i\in Q_0} kQe_i \otimes_k e_iM$$

equal to ϵ_α on component α.

Lemma 2.10. *We have a projective resolution*

$$0 \to \bigoplus_{\alpha\in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}M \xrightarrow{\partial-\epsilon} \bigoplus_{i\in Q_0} kQe_i \otimes_k e_iM \xrightarrow{\zeta} M \to 0$$

Proof. We will show that the sequence is a split short exact sequence of k vector spaces, which will be sufficient. Note that

$$M = \bigoplus_{i\in Q_0} e_iM$$

as a vector space. We have a map

$$s : M \to \bigoplus_{i\in Q_0} kQe_i \otimes_k e_iM$$

defined by

$$s(m) = \sum_{i \in Q_0} e_i \otimes e_i \cdot m$$

for $m \in M$. This satisfies $1_M = \zeta \circ s$. We also have a map

$$t : \bigoplus_{i \in Q_0} kQe_i \otimes_k e_i M \to \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha} M$$

defined by

$$t(\rho \otimes m) = \sum_{k=1}^{n} \rho_k \otimes \rho'_{k-1} \cdot m$$

where $\rho = \alpha_n \alpha_{n-1} \cdots \alpha_1$ is a path in Q, $\rho_k = \alpha_n \alpha_{n-1} \cdots \alpha_{k+1}$, $\rho'_k = \alpha_k \alpha_{k-1} \cdots \alpha_1$ and the term $\rho_k \otimes \rho'_{k-1} \cdot m$ lies in the component corresponding to α_k. Note that $\rho_n = e_{s\alpha_{k+1}}$ and $\rho'_0 = e_{s\alpha_1}$. Now by a simple calculation we get that

$$s \circ \zeta(\rho \otimes m) = e_i \otimes (\rho \cdot m)$$

where ρ is a path ending in vertex i, and

$$(\partial - \epsilon) \circ t(\rho \otimes m) = \rho \otimes m - e_i \otimes (\rho \cdot m)$$

and so

$$1 = (\partial - \epsilon) \circ t + s \circ \zeta$$

Also by a simple calculation $t \circ (\partial - \epsilon) = 1$, so the results follows. □

2.3 The main result

Now consider the module

$$M = \bigoplus_{\alpha \in Q_1} kQe_{s\alpha} \otimes_k e_{t\alpha} kQ \oplus \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes_k e_j kQ$$

Also let $\mathcal{Q}$ be the quiver we obtain from $\bar{Q}$ by adding an arrow $\beta_{ij} : j \to i$ for each pair of vertices i and j in $\bar{Q}$ with $i \neq j$.

Lemma 2.11. *There is a natural isomorphism*

$$T_{kQ} M \cong k\mathcal{Q}$$

of k algebras

Proof. Note that kQ sits inside $k\mathcal{Q}$ as a sub-k algebra, making $k\mathcal{Q}$ into a bi-kQ module. Now we have kQ bimodule morphisms

$$\Psi_\alpha : kQe_{s\alpha} \otimes_k e_{t\alpha} kQ \to k\mathcal{Q}$$

given by

$$\Psi_\alpha(\rho_0 \otimes \rho_1) = \rho_0\alpha^*\rho_1$$

Similarly we have a kQ bimodule morphism

$$\Psi_{i,j} : kQe_i \otimes e_jkQ \to k\mathcal{Q}$$

given by

$$\Psi_{i,j}(\rho_0 \otimes \rho_1) = \rho_0\beta_{ij}\rho_1$$

Taking the sum over all vertices $i \neq j$ and all arrows $\alpha \in Q_1$ we get kQ bimodule morphism

$$\Psi : M \to k\mathcal{Q}$$

By the universal property of the tensor algebra this induces a k-algebra morphism

$$\Psi : T_{kQ}M \to k\mathcal{Q}$$

which we will also write as Ψ. Note that Ψ must be surjective since its image is a k-algebra containing all the arrows of $\mathcal{Q}$. Now consider the k-module morphism

$$\Phi : k\mathcal{Q} \to T_{kQ}M$$

defined as follows. Let p be a path in $k\mathcal{Q}$. Then $p = p_0 \cdot q_0 \cdot p_1 \cdot q_1 \cdots q_{n-1} \cdot p_n$ for some n, where p_i is a path in Q, and q_j is an arrow β_{rs} or an arrow α^*. We then set

$$\Phi(p) = (p_0 \otimes_k p_1) \otimes_{kQ} (e_{sq_2} \otimes_k p_2) \otimes_{kQ} (e_{sq_3} \otimes_k p_3) \cdots \otimes_{kQ} (e_{sq_n} \otimes_k p_n)$$

where $e_{sq_i} \otimes_k p_i$ (resp $p_0 \otimes_k p_1$) lies in the component of M corresponding to α if $q_i = \alpha^*$ (resp $q_0 = \alpha^*$), and lies in the component of M corresponding to (i, j) if $q_i = \beta_{i,j}$ (resp $q_0 = \beta_{i,j}$) . Now a simple calculation gives that

$$\Phi \circ \Psi = 1$$

so Ψ is bijective and therefore an isomorphism of rings. □

We will need one more technical results

Lemma 2.12. *Let Q be a finite quiver without cycles, and let i and j be vertices of Q. Assume M is a kQ bimodule. Then there exists a natural isomorphism*

$$\mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_jkQ, M) \cong e_iMe_j$$

where e_iMe_j are the elemets in M satisfying $= e_i \cdot m = m = m \cdot e_j$.

Proof. We define a k morphism

$$\phi : \mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_j kQ, M) \to e_i M e_j$$

by

$$\phi(f) = f(e_i \otimes e_j)$$

Note that the map is well defined since

$$e_i \cdot f(e_i \otimes e_j) = f((e_i \cdot e_i) \otimes e_j) = f(e_i \otimes e_j)$$

and

$$f(e_i \otimes e_j) \cdot e_j = f(e_i \otimes e_j \cdot e_j) = f(e_i \otimes e_j)$$

We also have a map

$$\psi : e_i M e_j \to \mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_j kQ, M)$$

given by

$$\psi(m)(a \otimes b) = a \cdot m \cdot b$$

where $a \in kQe_i$ and $b \in e_j kQ$. Now a simple calculation shows that $\phi \circ \psi = 1$ and $\psi \circ \phi = 1$, so the result follows. $\square$

We can now finally prove the main result for this section (see also [3] and [17])

Theorem 2.13. *Let Q be a finite quiver without cycles. Then there exists an isomorphism*

$$\mathcal{P}_k(Q) \cong T_{kQ}(\theta)$$

which acts as identity on kQ, and which maps the arrows in Q_1^* into the augmentation ideal of $T_{kQ}(\theta)$, where $\theta = \mathrm{Ext}^1_{kQ}(D(kQ_{kQ}), kQ)$

Proof. We have an exact sequence

$$0 \to \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha} kQ \xrightarrow{\partial - \epsilon} \bigoplus_{i \in Q_0} kQe_i \otimes_k e_i kQ \xrightarrow{\zeta} kQ \to 0$$

as in lemma 2.10 with $M = kQ$. We also have the exact sequence

$$0 \to \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes e_j kQ \xrightarrow{1} \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes e_j kQ \to 0$$

Adding these two sequences together we get the exact sequence

$$0 \to \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}kQ \oplus \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes e_jkQ$$
$$\xrightarrow{g} \bigoplus_{i \in Q_0} kQe_i \otimes_k e_ikQ \oplus \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes e_jkQ \xrightarrow{f} kQ \to 0$$

where g is the sum of $\partial - \epsilon$ and 1, and f acts as ζ on the module $\bigoplus_{i \in Q_0} kQe_i \otimes_k e_ikQ$ and as 0 on $\bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes e_jkQ$. This exact sequences can be rewritten as

$$0 \to \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}kQ \oplus \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes e_jkQ \xrightarrow{g} kQ \otimes_k kQ \xrightarrow{m} kQ \to 0$$

where m is just the ordinary multiplication map. This implies in particular that

$$\Omega^1 kQ \cong \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}kQ \oplus \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes_k e_jkQ \tag{2.1}$$

Now observe that

$$\mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_jkQ, kQ \otimes_k kQ) \cong e_ikQ \otimes_k kQe_j \cong kQe_j \otimes_k e_ikQ$$

from lemma 2.12 and the fact that tensor product commute up to isomorphism. In fact the composition of these two isomorphisms will be an isomorphism of kQ^e modules. Note also that

$$\mathrm{Hom}_{kQ^e}(kQ, kQ \otimes_k kQ) \cong \mathrm{Der}(kQ, kQ \otimes_k kQ)$$

by proposition 2.2. Applying $\mathrm{Hom}_{kQ^e}(-, kQ \otimes_k kQ)$ to (2.1) we get

$$\mathrm{Der}(kQ, kQ \otimes_k kQ) \cong \bigoplus_{\alpha \in Q_1} kQe_{s\alpha} \otimes_k e_{t\alpha}kQ \oplus \bigoplus_{\substack{i \neq j \\ i,j \in Q_0}} kQe_i \otimes_k e_jkQ$$

Via the isomorphism in proposition 2.2 we see that $\Delta \in \mathrm{Der}(kQ, kQ \otimes_k kQ)$ correspond to the inclusion of $\Omega^1 kQ$ into $kQ \otimes_k kQ$ in $\mathrm{Hom}_{kQ^e}(\Omega^1 kQ, kQ \otimes_k kQ)$, which is just g. By using the isomorphism ϕ in lemma 2.12 we see that g corresponds to the element

$$\sum_{\alpha \in Q_1} (e_{s\alpha} \otimes \alpha - \alpha \otimes e_{t\alpha}) + \sum_{\substack{i \neq j \\ i,j \in Q_0}} e_i \otimes e_j$$

Now from lemma 2.11 the tensor algebra of $\mathrm{Der}(kQ, kQ \otimes_k kQ)$ is isomorphic to the path algebra $k\mathcal{Q}$. Under this isomorphism Δ is sent to

$$\delta = \sum_{\alpha \in Q_1} (\alpha^* \alpha - \alpha \alpha^*) + \sum_{\substack{i \neq j \\ i,j \in Q_0}} \beta_{ij}$$

in $k\mathcal{Q}$, and we have

$$(T_{kQ}\mathrm{Der}(kQ, kQ \otimes_k kQ))/(\Delta) \cong k\mathcal{Q}/(\delta)$$

Now for $i \neq j$ we get $e_i \delta e_j = \beta_{ij}$. This implies that

$$k\mathcal{Q}/(\delta) \cong \mathcal{P}_k(Q)$$

and the result follows from Corollary 2.9 $\square$

2.4 Applications

We will use Theorem 2.13 to show that the direct sum of all the indecomposable preprojective kQ modules is isomorphic to $\mathcal{P}_k(Q)$ as a left kQ module. This is a well known and important result. We will first mention some concepts used in the representation theory of finite dimensional algebra (see also [1]).
Let Λ will be a finite dimensional k algebra, where k is a field. $\mathrm{mod}\Lambda$ will denote the finitely generated right Λ modules. Recall that we have a functor

$$\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}(\Lambda)^{\mathrm{op}} \to (\mathrm{mod}\Lambda)^{\mathrm{op}}$$

which restricts to an equivalence

$$\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{proj}(\Lambda)^{\mathrm{op}} \to (\mathrm{proj}\Lambda)^{\mathrm{op}}$$

where $\mathrm{proj}\Lambda$ denotes the finitely generated projective kQ modules. The composition $\nu = D \circ \mathrm{Hom}_\Lambda(-, \Lambda)$, where D is the duality functor, is called the *Nakayama functor*. It's inverse is given by $\nu^{-1} = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-, \Lambda)$. The *Auslander-Reiten translation* τM of a finitely generated module M is defined as follows. First take a minimal projective presentation of M

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

Applying ν to it we get an exact sequence

$$\nu P_1 \xrightarrow{\nu f} \nu P_0 \to \nu M \to 0$$

We define $\tau M = \mathrm{Ker}(\nu f)$. Dually we can take a minimal injective presentation

$$0 \to M \to I_0 \xrightarrow{g} I_1$$

and define $\tau^{-}M = \operatorname{coker}(\nu^{-}g)$. We see that $\tau P = 0$ if and only if P is projective and $\tau^{-}I = 0$ if and only if I is injective. Furthermore it is not hard to see that if M is indecomposable non-projective then $\tau^{-}(\tau M) \cong M$ and if M is indecomposable non-injective then $\tau(\tau^{-}M) \cong M$. We say that a module M is *preprojective* if $\tau^{n}M = 0$ for some $n > 0$, and we say that M is *preinjective* if $\tau^{-n}M = 0$ for some $n > 0$
Now let $\Lambda = kQ$. Since kQ is hereditary, τ and τ^{-} extends to well defined functors on modkQ. We have the following lemma

Lemma 2.14. *There are natural isomorphism of functors*

1. $\tau^{-} \cong Ext^{1}_{kQ}(D(kQ), -)$
2. $\tau \cong Tor^{kQ}_{1}(-, D(kQ))$

Proof. We will prove (1), which is the only part we will need. Observe that $\nu^{-} = \operatorname{Hom}_{kQ^{\mathrm{op}}}(D(-), kQ) \cong \operatorname{Hom}_{kQ}(D(kQ), -)$ since D is contravariant and fully faithful. Also since kQ is hereditary, a minimal injective presentation of M will be an injective resolution of M. Hence applying ν^{-} to this resolution and taking the cokernel is the same as calculating $\operatorname{Ext}^{1}_{kQ}(D(kQ), -)$, so the result follows. □

We want to give another description of τ^{-}, but in order to do that we will need the following homological result. The proof can be found in most books on homological algebra.

Theorem 2.15. *Let R and S be rings, and let $F : modR \to modS$ be a right exact functor which preserve sums. Then*

$$F \cong F(R) \otimes_{R} -$$

Now let $\theta = \operatorname{Ext}^{1}_{kQ}(D(kQ), kQ)$ as before. We then have

Corollary 2.16. *There is a natural isomorphism*

$$\tau^{-} \cong \theta \otimes_{kQ} -$$

Proof. We know that $\tau^{-} \cong \operatorname{Ext}^{1}_{kQ}(D(kQ), -)$. Also $\operatorname{Ext}^{1}_{kQ}(D(kQ), -)$ preserve sums, and since kQ is hereditary $\operatorname{Ext}^{1}_{kQ}(D(kQ), -)$ is also right exact. The result follows from Theorem 2.15 □

Note that this imples that $\tau^{-n}(kQ) \cong \theta \otimes_{kQ} \theta \otimes_{kQ} \ldots \otimes_{kQ} \theta$, where the tensor product is taken n times. This gives us the following result

Corollary 2.17. *We have an isomorphism*

$$\mathcal{P}_{k}(Q) \cong \bigoplus_{n=0}^{\infty} \tau^{-n}(kQ)$$

as left kQ modules. In particular $\mathcal{P}_{k}(Q)$ is the direct sum of all the indecomposable preprojective modules.

Corollary 2.18. *$\mathcal{P}_k(Q)$ is finite dimensional if and only if the underlying graph of Q is of Dynkin type*

Proof. This follows from the well known result that there are finitely many indecomposable preprojective modules if and only if the underlying graph of Q is of Dynkin type. $\square$

3. DG ALGEBRAS AND TRIANGULATED CATEGORIES

In this section we introduce differential graded algebras and differential graded modules (see also [10]). Dg modules over a dg algebra is a generalization of a chain complex over a ring. A lot of the same constructions and results will also work in this case. We can for example define the tensor product of two dg modules. We also have a well defined homotopy category and derived category of a dg algebra. These will be triangulated in a similar way as for a ring.

We also consider differential graded categories , i.e categories enriched over chain complexes (see [8] and [9]). These are generalizations of differential graded algebras. In fact a dg algebra is precisely a dg category with one object. If A is a dg algebra we have the dg category $\mathcal{C}_{dg}(A)$ with objects being dg modules over A and with morphisms being graded maps. This will turn out to be a very important category for studying the dg modules over A

In this section k is a commutative ring.

3.1 Definitions and examples

A *differential graded k algebra* A is a chain complex which is also a graded k algebra such that the multiplication map

$$\text{mult} : A \otimes_k A \to A$$

is a morphism of chain complexes. It is not hard to see that this is equivalent to

$$d(a \cdot b) = da \cdot b + (-1)^i a \cdot db$$

for all $a \in A^i$ and $b \in A^j$ and for all i, j. A *morphism* of dg algebras A and B is a chain map

$$f : A \to B$$

satisfying

$$f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$$

for all $a_1, a_2 \in A$.

Example 3.1. If A is an ordinary k algebra, then A can be considered as a dg algebra concentrated in degree 0.

Example 3.2. Let M be a chain complex over k. Let $\mathcal{H}\mathrm{om}_k(M,M)^n$ be the module consisting of all k linear maps $f : M \to M$ of degree n, i.e $f(M^i) \subset M^{i+n}$. Note that f doesn't necessarily commute with the differential of M. Combining these modules gives us a chain complex

$$\mathcal{H}\mathrm{om}_k(M,M)$$

with degree n component equal to $\mathcal{H}\mathrm{om}_A(M,M)^n$ and with differential given by

$$d(f) = d_M \circ f - (-1)^n f \circ d_M$$

for $f \in \mathcal{H}\mathrm{om}_k(M,M)^n$. Now $\mathcal{H}\mathrm{om}_k(M,M)$ also has a natural ring structure where multiplication is given by composition. Since

$$d(f \circ g) = d_M \circ f \circ g - (-1)^{m+n} f \circ g \circ d_M$$

and

$$\begin{aligned}
&d(f) \circ g + (-1)^m f \circ d(g) \\
&= d_M \circ f \circ g - (-1)^m f \circ d_M \circ g + (-1)^m f \circ d_M \circ g - (-1)^{m+n} f \circ g \circ d_M \\
&= d_M \circ f \circ g - (-1)^{m+n} f \circ g \circ d_M
\end{aligned}$$

we see that $\mathcal{H}\mathrm{om}_k(M,M)$ is a dg k algebra.

Example 3.3. This example requires some knowledge of differential geometry. Let $\Omega^p(U)$ denote the $\mathbb{R}$-vector space of all alternating smooth p forms on U, where U is an open subset of $\mathbb{R}^n$. So for $x \in U$, and $\omega \in \Omega^p(U)$ we have a linear alternating map

$$\omega(x) : V \times V \times ... \times V \to \mathbb{R}$$

where we take the product p times with $V = \mathbb{R}^n$. Now $\omega(x)$ is alternating means that

$$\omega(x)(v_{\sigma(1)}, ...v_{\sigma(p)}) = \mathrm{sign}(\sigma) \cdot \omega(x)(v_1, ...v_p)$$

for $\sigma \in S_p$, where S_p is the group of permutations of $\{1, 2, ...p\}$, and we write $w(x) \in \mathrm{Alt}^p(\mathbb{R}^n)$. Collecting all these $\Omega^p(U)$ together we get a chain complex $\Omega(U)$ with differential

$$d : \Omega^p(U) \to \Omega^{p+1}(U)$$

given by

$$d\omega(x)(v_1, ...v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} D_x\omega(v_i)(v_1, ..., v_{i-1}, v_{i+1}, ..., v_{p+1})$$

where $D_x\omega$ is the induced map between the tangent spaces at x, i.e

$$D_x\omega : \mathbb{R}^n \to \mathrm{Alt}^p(\mathbb{R}^n)$$

Now $\Omega(U)$ also comes equipped with a multiplicative structure

$$\wedge : \omega^p(U) \times \omega^q(U) \to \omega^{p+q}(U)$$

defined by

$$\begin{aligned}&\omega_1 \wedge \omega_2(x)(v_1, ..., v_{p+q})\\&= \frac{1}{p!q!} \sum_{\sigma \in S(p+q)} \mathrm{sign}(\sigma)\omega_1(v_{\sigma(1)}, ...v_{\sigma(p)})\omega_2(v_{\sigma(p+1)}, ...v_{\sigma(p+q)})\end{aligned}$$

and together with the differential d this makes $\Omega(U)$ into a differential graded algebra over $\mathbb{R}$. In fact this example can be generalized to any manifold. Details of this construction is provided in chapter 9 of [16]

Let A be a dg algebra. We want to construct the opposite dg algebra A^{op} which has the same elements as A and with opposite multiplication to A. Naively defining

$$a * b = b \cdot a$$

won't work, since then

$$* : A \otimes_k A \to A$$

will not be a morphism of chain complexes. Instead we define $* = m \circ \tau$, i.e as the composition

$$A \otimes_k A \xrightarrow{\tau} A \otimes A \xrightarrow{m} A$$

where τ was defined (1.1) and m is the multiplication map in A. We immediately get that $*$ is a morphism of chain complexes, and hence A^{op} is a dg algebra. Explicitely $*$ is defined by

$$a * b = (-1)^{|a|\cdot|b|} \cdot b \cdot a$$

Observe that

$$\begin{aligned}d(a * b) &= (-1)^{|a|\cdot|b|} \cdot d(ba)\\&= (-1)^{|a|\cdot|b|} \cdot d(b) \cdot a + (-1)^{|a|\cdot|b|+|b|} \cdot b \cdot d(a)\end{aligned}$$

and

$$d(a) * b + (-1)^{|a|} \cdot a * d(b) = (-1)^{(|a|+1)\cdot|b|} \cdot b \cdot d(a) + (-1)^{|a|\cdot|b|} \cdot d(b) \cdot a$$

and hence

$$d(a * b) = d(a) * b + (-1)^{|a|} \cdot a * d(b)$$

which was expected since A^{op} is a dg algebra.
The idea of composing with τ in order to get a dg algebra structure or a dg module structure (defined below) will be used frequently in this thesis, but it might not always be stated explicitely.

3.2 Dg modules

Let A be a differential graded algebra. A *right differential graded module* over A is a chain complex M over k with a right A module structure such that the map

$$\text{mult} : M \otimes_k A \to M$$

given by

$$\text{mult}(m \otimes a) = m \cdot a$$

is a morphism of chain complexes. It is not hard to see that this is equivalent to

$$m \cdot a \in M^{i+j}$$

if $a \in A^i$ and $m \in M^j$, and

$$d(m \cdot a) = d(m) \cdot a + (-1)^{|m|} \cdot m \cdot d(a)$$

A *left differential graded module* over A is a chain complex M over k with a left A module structure such that the map

$$\text{mult} : A \otimes_k M \to M$$

given by

$$\text{mult}(a \otimes m) = a \cdot m$$

is a morphism of chain complexes. This is equivalent to

$$a \cdot m \in M^{i+j}$$

if $a \in A^i$ and $m \in M^j$, and

$$d(a \cdot m) = d(a) \cdot m + (-1)^{|a|} \cdot a \cdot d(m)$$

Note that M is a right dg A module iff it is a left dg A^{op} module. These module structures are related via

$$a \circ m = (-1)^{|a| \cdot |m|} m \cdot a$$

where $\cdot$ comes from the right action of A and $\circ$ comes from the left action of A^{op}.

A *morphism* of right (resp left) dg A modules M and N is a chain map

$$f : M \to N$$

satisfying $f(m \cdot a) = f(m) \cdot a$ (resp $f(a \cdot m) = a \cdot f(m)$). We will denote the category of right dg A modules for $C(A)$. The set of morphisms is denoted by

$\mathrm{Hom}_{C(A)}(M, N)$.
A degree -1 graded map

$$s : M \to N$$

is a k-linear map satisfying $s(M^i) \subseteq N^{i+1}$. Furthermore we have

$$s(a \cdot m) = (-1)^{|a|} \cdot a \cdot s(m)$$

if M and N are left modules and

$$s(m \cdot a) = s(m) \cdot a$$

if M and N are right modules. It does not necessarily commute with the differential. We will say more about such maps in section 3.4.
Two morphisms $f : M \to N$ and $g : M \to N$ are said to be *homotopic*, written

$$f \sim g$$

if there exists a degree -1 graded map

$$s : M \to N$$

satisfying

$$f - g = d_N \circ s + s \circ d_M$$

A morphism f is said to be *nullhomotopic* if $f \sim 0$. Also we say that M is *homotopic* to N and write $M \sim N$ if there exists morphisms $f : M \to N$ and $g : N \to M$ of dg A modules satisfying

$$g \circ f \sim 1_M \quad \text{and} \quad f \circ g \sim 1_N$$

The *homotopoy category* $\mathcal{H}(A)$ has the same objects as $C(A)$ and its morphisms $\mathrm{Hom}_{\mathcal{H}(A)}(M, N)$ are homotopy classes of morphisms in $C(A)$. Note that two modules are homotopic if and only if they are isomorphic in $\mathcal{H}(A)$.

3.3 Shifts and cones

Let M be a right (resp left) dg A module. The nth shift of M, denoted by $M[n]$, is the dg A module with components

$$M[n]^k = M^{k+n}$$

and differential $d^k_{M[n]} = (-1)^n \cdot d^{k+n}_M$. The right (resp left) action $*$ of A on $M[n]$ is given by $m * a = m \cdot a$ (resp $a * m = (-1)^{n \cdot |a|} \cdot a \cdot m$). If $f : M \to N$ is a morphism we let

$$f[n] : M[n] \to N[n]$$

be the morphism which is defined componentwise by $f[n]_k = f_{k+n}$. Then $[n]$ induces a automorphism

$$[n] : C(A) \to C(A)$$

Note that $[n] \circ [m] = [n+m]$.

Now let M and N be two right (resp left) dg modules over the dg algebra A, and let $u : M \to N$ be a morphism of dg modules. The cone of u, denoted by Cone(u), is the dg A module with components

$$\mathrm{Cone}(u)^i = N^i \oplus M^{i+1}$$

and differential

$$d_{\mathrm{Cone}(u)} = \begin{pmatrix} d_N & u \\ 0 & d_{M[1]} \end{pmatrix}$$

Hence $d_{\mathrm{Cone}(u)}(n,m) = (d_N(n) + u(m), d_{M[1]}(m))$. The right (resp left) A module structure on Cone(u) is given by $(n,m) \cdot a = (n \cdot a, m \cdot a)$ (resp $a \cdot (n,m) = ((-1)^{|a|} \cdot a \cdot n, a \cdot m)$). Let $(n,m) \in N^i \oplus M^{i+1}$. The calculation

$$\begin{aligned}
&d_{\mathrm{Cone}(u)}((n,m) \cdot a) = d_{\mathrm{Cone}(u)}(n \cdot a, m \cdot a) \\
&= (d(n \cdot a) + u(m \cdot a), -d(m \cdot a)) \\
&= (d(n) \cdot a + (-1)^i \cdot n \cdot d(a) + u(m) \cdot a, -d(m) \cdot a - (-1)^{i+1} \cdot m \cdot d(a)) \\
&= (d(n) \cdot a + u(m) \cdot a, -d(m) \cdot a) + ((-1)^i n \cdot d(a), -(-1)^{i+1} m \cdot d(a)) \\
&= (d(n) + u(m), -d(m)) \cdot a + (-1)^i (n,m) \cdot d(a) \\
&= d_{\mathrm{Cone}(u)}(n,m) \cdot a + (-1)^i (n,m) \cdot d(a)
\end{aligned}$$

shows that Cone(u) is a right dg module when M and N are right dg modules. A similar argument works when M and N are left dg modules. Observe that we have maps

$$v : N \to \mathrm{Cone}(u)$$

and

$$w : \mathrm{Cone}(u) \to M[1]$$

given by $v(n) = (n, 0)$ and $w(n,m) = m$. It is easy to see that these are morphisms of dg modules. We say that

$$M \xrightarrow{u} N \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} M[1]$$

is a *strict triangle* and denote it by (u, v, w). Note that we have an isomorphism

$$\mathrm{Cone}(u) \cong N \oplus M[1]$$

as graded modules (not as chain complexes).

Lemma 3.4. *Assume we have a commutative diagram*

$$\begin{tikzcd} M_1 \arrow[r, "u_1"] \arrow[d, "f"] & N_1 \arrow[d, "g"] \\ M_2 \arrow[r, "u_2"] & N_2 \end{tikzcd}$$

Let (u_1, v_1, w_1) *and* (u_2, v_2, w_2) *be the strict triangles corresponding to* u_1 *and* u_2*. Then the map* $h : \mathrm{Cone}(u_1) \to \mathrm{Cone}(u_2)$ *given by*

$$h = \begin{pmatrix} g & 0 \\ 0 & f[1] \end{pmatrix}$$

is a morphism of dg modules.

Proof. Note that

$$\begin{pmatrix} g & 0 \\ 0 & f[1] \end{pmatrix} \circ \begin{pmatrix} d_{N_1} & u \\ 0 & d_{M_1[1]} \end{pmatrix} = \begin{pmatrix} g \circ d_{N_1} & g \circ u \\ 0 & f[1] \circ d_{M_1[1]} \end{pmatrix}$$

and

$$\begin{pmatrix} d_{N_2} & u \\ 0 & d_{M_2[1]} \end{pmatrix} \circ \begin{pmatrix} g & 0 \\ 0 & f[1] \end{pmatrix} = \begin{pmatrix} d_{N_2} \circ g & u \circ f[1] \\ 0 & d_{M_2[1]} \circ f[1] \end{pmatrix}$$

are equal since f and g are chain maps satisfying $g \circ u = u \circ f$. Hence h is a chain map and the result follows. □

3.4 Hom of dg modules

Let M and N be two right (resp left) dg A-modules. Let $\mathcal{H}\mathrm{om}_A(M, N)^n$ be the k module consisting of all morphisms $f : M \to N$ of degree n satisfying

$$f(m \cdot a) = f(m) \cdot a$$
$$(\text{resp } f(a \cdot m) = (-1)^{n \cdot |a|} a \cdot f(m))$$

Observe that the degree -1 graded map in section 3.2 is just an element of $\mathcal{H}\mathrm{om}_A(M, N)^{-1}$. Combining the k modules $\mathcal{H}\mathrm{om}_A(M, N)^n$ gives us a chain complex

$$\mathcal{H}\mathrm{om}_A(M, N)$$

with degree n component equal to $\mathcal{H}\mathrm{om}_A(M, N)^n$ and with differential given by

$$d(f) = d_N \circ f - (-1)^{|f|} \cdot f \circ d_M$$

This construction is similar to the one in example 3.1. Not that f is a morphism if and only if $f \in \mathrm{Z}^0(\mathcal{H}\mathrm{om}_A(M,N))$ and $f \sim 0$ if and only if $f \in \mathrm{B}^0(\mathcal{H}\mathrm{om}_A(M,N))$. In particular we have that

$$\mathrm{Hom}_{\mathcal{H}A}(M,N) = \mathrm{H}^0(\mathcal{H}\mathrm{om}_A(M,N))$$

Now assume M and N are right dg A-modules. If M also has a left dg B module structure then $\mathcal{H}\mathrm{om}_A(M,N)$ gets a right B module structure via

$$f \cdot b = f \circ (b \cdot -)$$

for $f \in \mathcal{H}\mathrm{om}_A(M,N)$ and $b \in B$, where $b \cdot - : M \to M$ is left multiplication by b. Since

$$d(f \cdot b) = d(f \circ (b \cdot -)) = d \circ f \circ (b \cdot -) - (-1)^{|f|+|b|} \cdot f \circ (b \cdot -) \circ d$$

and

$$\begin{aligned} &d(f) \cdot b + (-1)^{|f|} \cdot f \cdot db \\ &= d \circ f \circ (b \cdot -) - (-1)^{|f|} \cdot f \circ d \circ (b \cdot -) + (-1)^{|f|} \cdot f \circ (db \cdot -) \end{aligned}$$

and

$$(-1)^{|f|} \cdot f \circ d \circ (b \cdot -) = (-1)^{|f|} \cdot f \circ (db \cdot -) + (-1)^{|f|+|b|} \cdot f \circ (b \cdot -) \circ d$$

we get that

$$d(f \cdot b) = d(f) \cdot b + (-1)^{|f|} \cdot f \cdot db$$

and hence $\mathcal{H}\mathrm{om}_A(M,N)$ is a right dg B module. On the other hand if N has a left dg B module structure then $\mathcal{H}\mathrm{om}_A(M,N)$ gets a left B module structure via the action

$$b \cdot f = (b \cdot -) \circ f$$

Since

$$d(b \cdot f) = d \circ (b \cdot -) \circ f - (-1)^{|b|+|f|} \cdot (b \cdot -) \circ f \circ d$$

and

$$d \circ (b \cdot -) \circ f = (db \cdot -) \circ f + (-1)^{|b|} \cdot (b \cdot -) \circ d \circ f$$

and

$$\begin{aligned} &d(b) \cdot f + (-1)^{|b|} \cdot b \cdot d(f) \\ &= (db \cdot -) \circ f + (-1)^{|b|} \cdot (b \cdot -) \circ d \circ f - (-1)^{|b|+|f|} \cdot (b \cdot -) \circ f \circ d \end{aligned}$$

we get that

$$d(b \cdot f) = d(b) \cdot f + (-1)^{|b|} \cdot b \cdot d(f)$$

so $\mathcal{H}\mathrm{om}_A(M,N)$ will be a left dg B module.
We can do a similar construction when M and N are left dg A-modules. If M has a right dg B module structure then $\mathcal{H}\mathrm{om}_A(M,N)$ gets a left dg B module structure given by

$$b \cdot f = (-1)^{|b|\cdot|f|} \cdot f \circ (b * -)$$

where $b * - : M \to M$ comes from the left action of B^{op} on M. Hence

$$(b \cdot f)(m) = (-1)^{|b|\cdot|f|+|b|\cdot|m|} \cdot f(m \cdot b)$$

Similarly if N has a right dg B module structure then $\mathcal{H}\mathrm{om}_A(M,N)$ will be a right dg B module via

$$f \cdot b = (-1)^{|f|\cdot|b|} \cdot (b * -) \circ f$$

where $b * - : N \to N$ comes from the left action of B^{op} on N. So

$$(f \cdot b)(m) = (-1)^{|b|\cdot|m|} \cdot f(m) \cdot b$$

Now consider the shift $M[n]$ of M. There is a graded map

$$\Sigma_M^n : M \to M[n]$$

acting as identity on the underlying modules. Note that

$$d_{M[n]} = (-1)^n \cdot \Sigma_M^n \circ d_M \circ \Sigma_{M[n]}^{-n}$$

and hence

$$\begin{aligned} d(\Sigma_M^n) &= d_{M[n]} \circ \Sigma_M^n - (-1)^n \cdot \Sigma_M^n \circ d_M \\ &= (-1)^n \cdot \Sigma_M^n \circ d_M - (-1)^n \cdot \Sigma_M^n \circ d_M = 0 \end{aligned}$$

This implies that $\Sigma_M^n \in Z^{-n}\mathcal{H}\mathrm{om}_A(M, M[n])$.

Lemma 3.5. *Let M and N both be left or right dg A-modules.*

1. *There is a natural isomorphism of chain complexes*

$$\mathcal{H}\mathrm{om}_A(M, N[n]) \cong \mathcal{H}\mathrm{om}_A(M,N)[n]$$

   sending $\Sigma_N^n \circ f$ to $\Sigma_{\mathcal{H}\mathrm{om}_A(M,N)}^n(f)$

2. *There is a natural isomorphism of chain complexes*

$$\mathcal{H}\mathrm{om}_A(M[n], N) \cong \mathcal{H}\mathrm{om}_A(M,N)[-n]$$

   sending $f \circ \Sigma_{M[n]}^{-n}$ to $(-1)^{n\cdot(|f|+n)} \cdot \Sigma_{\mathcal{H}\mathrm{om}_A(M,N)}^{-n}(f)$

Proof. We will only show the second part. The argument for the first part is similar. Let

$$\phi^n : \mathcal{H}\mathrm{om}_A(M[n], N) \to \mathcal{H}\mathrm{om}_A(M, N)[-n]$$

denote the map. Observe that

$$\begin{aligned}\phi^n(d_{\mathcal{H}\mathrm{om}_A(M[n],N)}(f \circ \Sigma^{-n}_{M[n]})) &= \phi^n(d_{\mathcal{H}\mathrm{om}_A(M,N)}(f) \circ \Sigma^{-n}_{M[n]}) \\ &= (-1)^{n\cdot(|f|+1+n)} \cdot \Sigma^{-n}_{\mathcal{H}\mathrm{om}_A(M,N)}(d_{\mathcal{H}\mathrm{om}_A(M,N)}(f))\end{aligned}$$

since $d(\Sigma^{-n}_{M[n]}) = 0$. Also

$$\begin{aligned}d_{\mathcal{H}\mathrm{om}_A(M,N)[-n]}(\phi^n(f \circ \Sigma^{-n}_{M[n]})) &= (-1)^{n\cdot(|f|+n)} \cdot d_{\mathcal{H}\mathrm{om}_A(M,N)[-n]}(\Sigma^{-n}_{\mathcal{H}\mathrm{om}_A(M,N)}(f)) \\ &= (-1)^{n\cdot(|f|+1+n)} \cdot \Sigma^{-n}_{\mathcal{H}\mathrm{om}_A(M,N)}(d_{\mathcal{H}\mathrm{om}_A(M,N)}(f))\end{aligned}$$

This implies that ϕ^n is a morphism of chain complexes, so the result follows. $\square$

We use the same notation as in the proof, i.e the isomorphism in part 2 of the lemma is given by

$$\phi^n : \mathcal{H}\mathrm{om}_A(M[n], N) \cong \mathcal{H}\mathrm{om}_A(M, N)[-n]$$

Now consider the diagram

$$\begin{tikzcd} \mathcal{H}\mathrm{om}_A(M[n+m], N) \arrow[r, "\phi^{n+m}"] \arrow[d, "\phi^m"] & \mathcal{H}\mathrm{om}_A(M, N)[-m-n] \arrow[d, "="] \\ \mathcal{H}\mathrm{om}_A(M[n], N)[-m] \arrow[r, "{[-m](\phi^n)}"] & \mathcal{H}\mathrm{om}_A(M, N)[-m-n] \end{tikzcd}$$

A simple calculation gives us that

$$\phi^{n+m}(f \circ \Sigma^{-n-m}_{M[n+m]}) = (-1)^{(n+m)\cdot(|f|+n+m)} \cdot \Sigma^{-n-m}_{\mathcal{H}\mathrm{om}_A(M,N)}(f)$$

and

$$[-m](\phi^n) \circ \phi^m(f \circ \Sigma^{-n-m}_{M[n+m]}) = (-1)^{n\cdot(|f|+n)+m\cdot(|f|+n+m)} \cdot \Sigma^{-n-m}_{\mathcal{H}\mathrm{om}_A(M,N)}(f)$$

Observe that the diagram doesn't commute because of an exstra sign $(-1)^{n\cdot m}$. We rectify this by defining a new isomorphism

$$(-1)^{n(n+1)/2} \cdot \phi^n : \mathcal{H}\mathrm{om}_A(M[n], N) \cong \mathcal{H}\mathrm{om}_A(M, N)[-n]$$

This gives us the following result.

Lemma 3.6. *There is a natural isomorphism of chain complexes*

$$\mathcal{H}\mathrm{om}_A(M[n], N) \cong \mathcal{H}\mathrm{om}_A(M, N)[-n]$$

sending $f \circ \Sigma^{-n}_{M[n]}$ *to* $(-1)^{n(n-1)/2+n\cdot(|f|+n)} \cdot \Sigma^{-n}_{\mathcal{H}\mathrm{om}_A(M,N)}(f)$*. This isomorphism makes the diagram*

$$\begin{tikzcd}
\mathcal{H}\mathrm{om}_A(M[n+m], N) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathcal{H}\mathrm{om}_A(M, N)[-m-n] \arrow[d, "="] \\
\mathcal{H}\mathrm{om}_A(M[n], N)[-m] \arrow[r, "\cong"] & \mathcal{H}\mathrm{om}_A(M, N)[-m-n]
\end{tikzcd}$$

commute.

3.5 Tensor product of dg modules

Let M be a right dg A-module, and N a left dg A-module. Since they are both chain complexes over k, we can construct the chain complex $M \otimes_k N$. Consider the k submodule P of $M \otimes_k N$ generated by elements of the form

$$m \cdot a \otimes n - m \otimes a \cdot n$$

Note that P is a graded submodule of $M \otimes_k N$. Since

$$\begin{aligned}
&d(m \cdot a \otimes n - m \otimes a \cdot n) = d(m \cdot a \otimes n) - d(m \otimes a \cdot n) \\
&= d(m) \cdot a \otimes n + (-1)^{|m|} \cdot m \cdot d(a) \otimes n + (-1)^{|m|+|a|} \cdot m \cdot a \otimes d(n) \\
&- d(m) \otimes a \cdot n - (-1)^{|m|} \cdot m \otimes d(a) \cdot n - (-1)^{|m|+|a|} \cdot m \otimes a \cdot d(n) \\
&= (d(m) \cdot a \otimes n - d(m) \otimes a \cdot n) + (-1)^{|m|} \cdot (m \cdot d(a) \otimes n - m \otimes d(a) \otimes n) \\
&+ (-1)^{|m|+|a|} \cdot (m \cdot a \otimes d(n) - m \otimes a \cdot d(n))
\end{aligned}$$

we get that P must be a chain complex. If we let

$$M \otimes_A N$$

denote the quotient of $M \otimes_k N$ by P, we get that $M \otimes_A N$ is a chain complex over k with differential inherited from $M \otimes_k N$. In particular we have that

$$m \cdot a \otimes n = m \otimes a \cdot n$$

and

$$d(m \otimes n) = d(m) \otimes n + (-1)^i m \otimes d(n)$$

holds in $M \otimes_A N$. If M also has a left dg B module structure, then $M \otimes_A N$ has a left B module structure given by

$$b \cdot (m \otimes n) = (b \cdot m) \otimes n$$

Since

$$\begin{aligned}
&d(b \cdot (m \otimes n)) = d((b \cdot m) \otimes n) \\
&d(b \cdot m) \otimes n + (-1)^{|b|+|m|} \cdot (b \cdot m) \otimes d(n) \\
&= (d(b) \cdot m) \otimes n + (-1)^{|b|} \cdot (b \cdot d(m)) \otimes n + (-1)^{|b|+|m|} \cdot (b \cdot m) \otimes d(n) \\
&= (d(b) \cdot m) \otimes n + (-1)^{|b|} \cdot b \cdot (d(m) \otimes n + (-1)^{|m|} \cdot m \otimes d(n)) \\
&= d(b) \cdot (m \otimes n) + (-1)^{|m|} \cdot b \cdot d(m \otimes n)
\end{aligned}$$

we get that $M \otimes_A N$ is a left dg B module. If M instead has a right dg B module structure then $M \otimes_A N$ has a right dg B module structure given by

$$(m \otimes n) \cdot b = (-1)^{|b| \cdot |n|} \cdot (m \cdot b) \otimes n$$

This can be shown in a similar way as above.
We have similiar constructions for N. If N has a left dg B module structure then $M \otimes_A N$ has a left dg B module structure given by

$$b \cdot (m \otimes n) = (-1)^{|b| \cdot |m|} \cdot m \otimes (b \cdot n)$$

and if N has a right dg B module structure then $M \otimes_A N$ has a right dg B module structure given by

$$(m \otimes n) \cdot b = m \otimes (n \cdot b)$$

We have the following result relating the shift functor and the tensor product.

Lemma 3.7. *Let M be a right dg A-module and N a left dg A-module.*

1. *There is a natural isomorphism of chain complexes*

$$M[k] \otimes_A N \cong (M \otimes_A N)[k]$$

   sending $\Sigma^k_M(m) \otimes n$ to $\Sigma^k_{M \otimes N}(m \otimes n)$

2. *There is a natural isomorphism of chain complexes*

$$M \otimes_A N[k] \cong (M \otimes_A N)[k]$$

   sending $m \otimes \Sigma^k_N(n)$ to $(-1)^{k \cdot |m|} \cdot \Sigma^k_{M \otimes N}(m \otimes n)$

Proof. We will only prove part 2. The proof of part 1 is similar. Let

$$\phi : M \otimes_A N[k] \to (M \otimes_A N)[k]$$

denote the map. We first need to show that ϕ is well defined. This holds since

$$\begin{aligned}
\phi(m \otimes a \cdot \Sigma^k_N(n)) &= (-1)^{k \cdot |a|} \cdot \phi(m \otimes \Sigma^k_N(a \cdot n)) \\
&= (-1)^{k \cdot (|a|+|m|)} \cdot \Sigma^k_{M \otimes N}(m \otimes a \cdot n)
\end{aligned}$$

and

$$\begin{aligned}\phi(m \cdot a \otimes \Sigma_N^k(n)) &= (-1)^{k\cdot(|a|+|m|)} \cdot \Sigma_{M\otimes N}^k(m \cdot a \otimes n) \\ &= (-1)^{k\cdot(|a|+|m|)} \cdot \Sigma_{M\otimes N}^k(m \otimes a \cdot n)\end{aligned}$$

We also have that

$$\begin{aligned}&\phi(d(m \otimes \Sigma_N^k(n))) = \phi(d_M(m) \otimes \Sigma_N^k(n)) + (-1)^{|m|} \cdot \phi(m \otimes d_{N[k]}(\Sigma_N^k(n))) \\ &= \phi(d_M(m) \otimes \Sigma_N^k(n)) + (-1)^{|m|+k} \cdot \phi(m \otimes \Sigma_N^k(d_N(n))) \\ &= (-1)^{k\cdot(m+1)} \cdot \Sigma_{M\otimes N}^k(d_M(m) \otimes n) + (-1)^{|m|+k+|m|\cdot k} \cdot \Sigma_{M\otimes N}^k(m \otimes d_N(n))\end{aligned}$$

and

$$\begin{aligned}&d(\phi(m \otimes \Sigma_N^k(n))) = (-1)^{k\cdot|m|} \cdot d(\Sigma_{M\otimes N}^k(m \otimes n)) \\ &= (-1)^{k\cdot(|m|+1)} \cdot \Sigma_{M\otimes N}^k(d(m \otimes n)) \\ &= (-1)^{k\cdot(|m|+1)} \cdot \Sigma_{M\otimes N}^k(d_M(m) \otimes n) + (-1)^{|m|+k+|m|\cdot k} \cdot \Sigma_{M\otimes N}^k(m \otimes d_N(n))\end{aligned}$$

hence they are equal and ϕ is an isomorphism of chain complexes. □

3.6 Differential graded categories

A *differential graded category* $\mathcal{C}$ is a category enriched over chain complexes. This means that the morphism spaces $\mathcal{C}(X, Y)$ are chain complexes over k and the composition

$$\circ : \mathcal{C}(Y, Z) \otimes_k \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

is a morphism of chain complexes. Note that there are categories $Z^0(\mathcal{C})$ and $H^0(\mathcal{C})$ associated to $\mathcal{C}$. They have the same objects as $\mathcal{C}$, and their morphism spaces are $Z^0(\mathcal{C}(X, Y))$ and $H^0(\mathcal{C}(X, Y))$ respectively. Note also that a differential graded category with one object is just a differential graded algebra. Now let $\mathcal{C}^{\text{op}}$ be the category with same objects as $\mathcal{C}$ and morphism spaces $\mathcal{C}^{\text{op}}(X, Y) = \mathcal{C}(Y, X)$. The composition

$$* : \mathcal{C}^{\text{op}}(Y, Z) \otimes_k \mathcal{C}^{\text{op}}(X, Y) \to \mathcal{C}^{\text{op}}(X, Z)$$

is given by $f * g = (-1)^{|f|\cdot|g|} \cdot g \circ f$. This makes $\mathcal{C}^{\text{op}}$ into a dg category.
Now assume that $\mathcal{C}$ and $\mathcal{D}$ are differential graded categories. A *differential graded functor*

$$F : \mathcal{C} \to \mathcal{D}$$

from $\mathcal{C}$ to $\mathcal{D}$ is a mapping that

- Associate to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$.
- Associate to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ such that

- $F(1_X) = 1_{F(X)}$ for all objects X in $\mathcal{C}$.
- $F(f \circ g) = F(f) \circ F(g)$ for all morphisms f, g in $\mathcal{C}$ such that the composition $f \circ g$ makes sense
- The map

$$F(X,Y) : \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$$

  is a morphism of chain complexes.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be dg functors. We have a complex

$$\mathcal{H}\text{om}(F,G)$$

with nth component $\mathcal{H}\text{om}(F,G)^n$ consisting of families of morphisms

$$\phi_X \in \mathcal{D}(FX,GX)^n$$

satisfying $\phi_Y \circ Ff = (-1)^{n\cdot|f|} \cdot Gf \circ \phi_X$. These are natural transformations in the context of dg categories. The differential on $\mathcal{H}\text{om}(F,G)$ is defined componentwise, i.e

$$d_{\mathcal{H}\text{om}(F,G)}(\phi)_X = d_{\mathcal{D}(FX,GX)}(\phi_X)$$

It is also not hard to see that if $\phi \in \mathcal{H}\text{om}(F,G)$ and $\psi \in \mathcal{H}\text{om}(G,H)$ then $\psi \circ \phi \in \mathcal{H}\text{om}(F,H)$ where

$$(\psi \circ \phi)_X = \psi_X \circ \phi_X$$

This gives a well defined morphism of chain complexes

$$\circ : \mathcal{H}\text{om}(G,H) \otimes_k \mathcal{H}\text{om}(F,G) \to \mathcal{H}\text{om}(F,H)$$

3.7 The dg category of dg modules

We want to show that the collection of right dg A-modules together with complexes of morphisms $\mathcal{H}\text{om}_A(M,N)$ gives us a dg category. In order to do that we need to show that composition

$$\circ : \mathcal{H}\text{om}_A(N,K) \otimes_k \mathcal{H}\text{om}_A(M,N) \to \mathcal{H}\text{om}_A(M,K)$$

sending $f \otimes g$ to $f \circ g$ is a morphism of chain complexes. This follows from

$$d(f \circ g) = d_K \circ f \circ g - (-1)^{|f|+|g|} f \circ g \circ d_M$$

and

$$\begin{aligned}
&d(f) \circ g + (-1)^{|f|} \cdot f \circ d(g) \\
&= d_K \circ f \circ g - (-1)^{|f|} \cdot f \circ d_N \circ g + (-1)^{|f|} \cdot (f \circ d_N \circ g - (-1)^{|g|} \cdot f \circ g \circ d_M) \\
&= d_K \circ f \circ g - (-1)^{|f|+|g|} \cdot f \circ g \circ d_M
\end{aligned}$$

We will denote this dg catgory by $\mathcal{C}_{dg}(A)$. It is easy to see that

$$Z^0(\mathcal{C}_{dg}(A)) = \mathcal{C}(A)$$

and

$$H^0(\mathcal{C}_{dg}(A)) = \mathcal{H}(A)$$

We say that a map $f : X \to Y$ in $\mathcal{C}_{dg}(A)$ is *closed* if

$$f \in Z^0(\mathcal{H}\mathrm{om}_A(X, Y)) = \mathrm{Hom}_{C(A)}(X, Y)$$

We can also extend the shift functor $[n] : C(A) \to C(A)$ to $\mathcal{C}_{dg}(A)$. It will act the same on objects, and send a mmap $f \in \mathcal{H}\mathrm{om}_A(M, N)$ to

$$[n](f) = (-1)^{n\cdot|f|} \cdot \Sigma^n_N \circ f \circ \Sigma^{-n}_{M[n]}$$

It is easy to see that this makes

$$[n] : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(A)$$

into a dg functor. Note that $[n] \circ [m] = [n + m]$ in $\mathcal{C}_{dg}(A)$. Note also that (cf section 3.6))

$$\Sigma^n \in Z^{-n}(\mathcal{H}\mathrm{om}(1_{\mathcal{C}_{dg}(A)}, [n]))$$

Now consider a componentwise split exact sequence in $\mathcal{C}_{dg}(A)$, i.e an exact sequence in $Z^0(\mathcal{C}_{dg}(A)) = \mathcal{C}(A)$

$$K \xrightarrow{f} M \xrightarrow{g} N$$

together with maps $s \in \mathcal{H}\mathrm{om}_A(N, M)^0$ and $t \in \mathcal{H}\mathrm{om}_A(M, K)^0$ satisfying

$$t \circ s = 0$$

and

$$g \circ s = 1_N$$

and

$$t \circ f = 1_K$$

We say that M is an *extension* of K and N. We then get a natural map $h \in \mathcal{H}\mathrm{om}_A(N[-1], K)^0$ given by

$$h = t \circ d(s) \circ \Sigma^1_{N[-1]} = -d(t) \circ s \circ \Sigma^1_{N[-1]} = t \circ d_Y \circ s \circ \Sigma^1_{N[-1]}$$

Lemma 3.8. *We use the same notation as above. Then*

$$h \in \mathrm{Hom}_{C(A)}(N[-1], M)$$

and we have an isomorphism

$$\mathrm{Cone}(h) \cong Y$$

making the diagram

$$\begin{tikzcd} N[-1] \arrow[r, "h"] & K \arrow[r, "v"] \arrow[dr, "f"'] & \mathrm{Cone}(h) \arrow[r, "w"] \arrow[d, "\cong"] & N \\ & & Z \arrow[ur, "g"'] & \end{tikzcd}$$

commute, where v and w are the maps defined in section 3.3.

Proof. Similar as for chain complexes over a ring □

Hence we get a one to one corresondence between componentwise split exact sequences in $\mathcal{C}_{dg}(A)$ and the cone of some morphism in $C(A)$.

Definition 3.9. Let strictperf(A) be the smallest full subcategory of $\mathcal{C}_{dg}(A)$ closed under shifts, extensions, and direct summands. A dg module M is called *strictly perfect* if it is an object of strictperf(A).

If A is an ordinary algebra then a bounded complex of finitely generated projectives is strictly perfect. Also if

$$F : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(B)$$

is a dg functor (covariant or contravariant) and if $F(A)$ is strictly perfect, then F takes strictly perfect modules to strictly perfect modules. This follows since F preserves extensions and commutes with the shift (which we will show below).

Now let M be a right dg A module and a left dg B module such that

$$(b \cdot m) \cdot a = b \cdot (m \cdot a)$$

for $a \in A$, $b \in B$ and $m \in M$.

Lemma 3.10. *We have a well defined dg functor*

$$\mathcal{H}\mathrm{om}_A(-, M) : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(B^{\mathrm{op}})^{\mathrm{op}}$$

sending N to

$$\mathcal{H}\mathrm{om}_A(N, M)$$

and a map $f \in \mathcal{H}\mathrm{om}_A(N_1, N_2)$ *to*

$$\mathcal{H}\mathrm{om}_A(f, M) : \mathcal{H}\mathrm{om}_A(N_2, M) \to \mathcal{H}\mathrm{om}_A(N_1, M)$$

given by

$$\mathcal{H}\mathrm{om}_A(f, M)(g) = (-1)^{|f|\cdot|g|} \cdot g \circ f$$

Proof. If N is a right dg A-module then $\mathcal{H}\mathrm{om}_A(N, M)$ has a left dg B-module structure given by

$$(b \cdot f)(x) = b \cdot f(x)$$

(cf section 3.4). Let N_1 and N_2 be dg A-modules and let $f \in \mathcal{H}\mathrm{om}_A(N_1, N_2)^n$. It is not hard to see that

$$\mathcal{H}\mathrm{om}_A(f, M) \in \mathcal{H}\mathrm{om}_{B^{\mathrm{op}}}(\mathcal{H}\mathrm{om}_A(N_2, M), \mathcal{H}\mathrm{om}_A(N_1, M))^n$$

We also have that

$$\begin{aligned}
&d(\mathcal{H}\mathrm{om}_A(f, M))(g) \\
&= d_{\mathcal{H}\mathrm{om}_A(N_1,M)} \circ \mathcal{H}\mathrm{om}_A(f, M)(g) - (-1)^{|f|} \cdot \mathcal{H}\mathrm{om}_A(f, M) \circ d_{\mathcal{H}\mathrm{om}_A(N_2,M)}(g) \\
&= (-1)^{|f|\cdot|g|} \cdot d_{\mathcal{H}\mathrm{om}_A(N_1,M)}(g \circ f) - (-1)^{|f|\cdot|g|} \cdot d_{\mathcal{H}\mathrm{om}_A(N_2,M)}(g) \circ f \\
&= (-1)^{|f|\cdot|g|+|g|} \cdot g \circ d_{\mathcal{H}\mathrm{om}_A(N_1,N_2)}(f) \\
&= \mathcal{H}\mathrm{om}_A(d_{\mathcal{H}\mathrm{om}_A(N_1,N_2)}(f), M)(g)
\end{aligned}$$

and hence $\mathcal{H}\mathrm{om}_A(-, M)$ is a dg functor. □

Lemma 3.11. *We have a well defined dg functor*

$$- \otimes_B M : \mathcal{C}_{dg}(B) \to \mathcal{C}_{dg}(A)$$

sending a right dg B*-module* N *to*

$$N \otimes_B M$$

and a map $f \in \mathcal{H}\mathrm{om}_A(N_1, N_2)$ *to*

$$f \otimes 1 : N_1 \otimes_B M \to N_2 \otimes_B M$$

given by

$$(f \otimes 1)(n_1 \otimes m) = f(n_1) \otimes m$$

Proof. This is a straightforward verification □

We now assume we have a dg functor

$$F : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(B)^{\mathrm{op}}$$

where A and B are dg algebras. Consider the dg natural transformations

$$\Sigma^{-n}_{[n]\circ F} \in Z^n(\mathcal{H}\mathrm{om}([n] \circ F, F))$$

and

$$F(\Sigma^n_{[-n]}) \in Z^{-n}(\mathcal{H}\mathrm{om}(F, F \circ [-n])$$

Let

$$\alpha^n = (-1)^{n(n+1)/2} \cdot F(\Sigma^n_{[-n]}) \circ \Sigma^{-n}_{[n]\circ F} \in Z^0(\mathcal{H}\mathrm{om}([n] \circ F, F \circ [-n]))$$

Note that α^n are natural isomorphisms in $C(A)$ and that $\alpha^0 = 1_F$. Also observe that

$$[n](\alpha^{-n}_{[-n]}) = (\alpha^n)^{-1} = (-1)^{n(n-1)/2} \cdot \Sigma^n_F \circ F(\Sigma^{-n})$$

(cf lemma 3.6). The next result tells us that the order of composition of the α^k doesn't matter.

Proposition 3.12. *Let α^n be as defined above. We have a commutative diagram*

$$\begin{tikzcd}
{[n] \circ [m] \circ F} \arrow[rr, "="] \arrow[d, "{[n]\alpha^m}"] & & {[n+m] \circ F} \arrow[d, "\alpha^{n+m}"] \\
{[n] \circ F \circ [-m]} \arrow[r, "\alpha^n_{[-m]}"] & {F \circ [-n] \circ [-m]} \arrow[r, "="] & {F \circ [-n-m]}
\end{tikzcd} \tag{3.1}$$

Proof. We have that

$$\begin{aligned}
\alpha^{n+m} &= (-1)^{(n+m)(n+m+1)/2} \cdot F(\Sigma^{n+m}_{[-n-m]}) \circ \Sigma^{-n-m}_{[n+m]\circ F} \\
[n]\alpha^m &= (-1)^{m(m+1)/2} \cdot \Sigma^n_{F\circ[-m]} \circ F(\Sigma^m_{[-m]}) \circ \Sigma^{-m}_{[m]\circ F} \circ \Sigma^{-n}_{[n]\circ[m]\circ F} \\
\alpha^n_{[-m]} &= (-1)^{n(n+1)/2} \cdot F(\Sigma^n_{[-n]\circ[-m]}) \circ \Sigma^{-n}_{[n]\circ F\circ[-m]}
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\alpha^n_{[-m]} \circ [n]\alpha^m \\
&= (-1)^{n(n+1)/2+m(m+1)/2} \cdot F(\Sigma^n_{[-n]\circ[-m]}) \circ F(\Sigma^m_{[-m]}) \circ \Sigma^{-m}_{[m]\circ F} \circ \Sigma^{-n}_{[n]\circ[m]\circ F} \\
&= (-1)^{n(n+1)/2+m(m+1)/2+m\cdot n} \cdot F(\Sigma^m_{[-m]} \circ \Sigma^n_{[-n]\circ[-m]}) \circ \Sigma^{-n-m}_{[n+m]\circ F} \\
&= (-1)^{(n+m)(n+m+1)/2} \cdot F(\Sigma^{n+m}_{[-n-m]}) \circ \Sigma^{-n-m}_{[n+m]\circ F} \\
&= \alpha^{n+m}
\end{aligned}$$

and the result follows. □

We also have the following result relating the cone and F.

Lemma 3.13. *Consider the sequence*

$$X \xrightarrow{u} Y \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} X[1]$$

with v and w the inclusion and projection respctively. We have extensions

$$F(X[1]) \xrightarrow{F(w)} F(\mathrm{Cone}(u)) \xrightarrow{F(v)} FY$$

and

$$F(X[1])[1] \xrightarrow{F(w)[1]} F(\mathrm{Cone}(u))[1] \xrightarrow{F(v)[1]} FY[1]$$

Hence we can write

$$F(\mathrm{Cone}(u)) = F(X[1]) \oplus FY$$

and

$$F(\mathrm{Cone}(u))[1] = F(X[1])[1] \oplus FY[1]$$

as graded modules. With this identification we get the following results

- *We have an isomorphism $f : F(\mathrm{Cone}(u)) \cong \mathrm{Cone}(-F(u[1]))$ given by*

$$f = \begin{pmatrix} 1 & 0 \\ 0 & (\alpha_Y^{-1})[1] \end{pmatrix}$$

  Hence the diagram

$$\begin{tikzcd} F(X[1]) \arrow[r, "F(w)"] \arrow[d, "="] & F(\mathrm{Cone}(u)) \arrow[r, "F(v)"] \arrow[d, "f"] & FY \arrow[d, "(\alpha_Y^{-1})[1]"] \\ F(X[1]) \arrow[r, "v_1"] & \mathrm{Cone}(-F(u[1])) \arrow[r, "w_1"] & F(Y[1])[1] \end{tikzcd}$$

  commutes, where v_1 and w_1 are the inclusion and projection respectively.

- *We have an isomorphism $g : \mathrm{Cone}(-Fu) \cong F(\mathrm{Cone}(u))[1]$ given by*

$$g = \begin{pmatrix} (\alpha_X^{-1})[1] & 0 \\ 0 & 1 \end{pmatrix}$$

  Hence the diagram

$$\begin{tikzcd} F(X) \arrow[r, "v_2"] \arrow[d, "(\alpha_X^{-1})[1]"] & \mathrm{Cone}(-F(u)) \arrow[r, "w_2"] \arrow[d, "g"] & (FY)[1] \arrow[d, "="] \\ F(X[1])[1] \arrow[r, "F(w)[1]"] & F(\mathrm{Cone}(u))[1] \arrow[r, "F(v)[1]"] & (FY)[1] \end{tikzcd}$$

  commute, where v_2 and w_2 are the inclusion and projection respectively.

Proof. Let $s : X[1] \to \mathrm{Cone}(u)$ be the inclusion and $t : \mathrm{Cone}(u) \to Y$ the projection. We know from lemma 3.8 that $u = t \circ d(s) \circ \Sigma^1_X$. Therefore

$$-u[1] = -\Sigma^1_Y \circ t \circ d(s)$$

and hence

$$-F(u[1]) = d(F(s)) \circ Ft \circ F(\Sigma^1_Y)$$

Applying F to the extension

$$Y \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} X[1]$$

gives us an extension

$$F(X[1]) \xrightarrow{Fw} F(\mathrm{Cone}(u)) \xrightarrow{Fv} FY$$

with splitting $Ft : FY \to F(\mathrm{Cone}(u))$ and $Fs : F(\mathrm{Cone}(u)) \to F(X[1])$. Using the isomorphism

$$\alpha_Y^{-1} = F(\Sigma^{-1}_{Y[1]}) \circ \Sigma^1_{F(Y)[-1]} : F(Y)[-1] \xrightarrow{\cong} F(Y[1])$$

we get a commutative diagram

$$\begin{tikzcd}
F(Y)[-1] \arrow[r, "-d(F(s)) \circ Ft \circ \Sigma^1_{F(Y)[-1]}"] \arrow[d, "\alpha_Y^{-1}"] & F(X[1]) \arrow[d, "="] \\
F(Y[1]) \arrow[r, "-F(u[1])"] & F(X[1])
\end{tikzcd}$$

and lemma 3.4 and 3.8 gives us an isomorphism $F(\mathrm{Cone}(u)) \cong \mathrm{Cone}(-F(u[1]))$ with the necessary properties.
For the second part we first observe that given a morphism $f : X_1 \to X_2$ we have an isomorphism

$$\mathrm{Cone}(f[-1]) \cong \mathrm{Cone}(f)[-1]$$

such that

$$\begin{tikzcd}
X_2[-1] \arrow[r, "\mathrm{incl}"] \arrow[d, "="] & \mathrm{Cone}(f[-1]) \arrow[r, "\mathrm{proj}"] \arrow[d, "\cong"] & X_1[-1] \arrow[d, "="] \\
X_2[-1] \arrow[r, "i[-1]"] & \mathrm{Cone}(f)[-1] \arrow[r, "j[-1]"] & X_1[-1]
\end{tikzcd}$$

commute, where $X_2 \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{j} X_1$ is the extension corresponding to the cone of f. Using this and our previous result we get an isomorphism

$$\mathrm{Cone}(-Fu) \cong F(\mathrm{Cone}(u[-1])) \cong F(\mathrm{Cone}(u)[-1]) \cong F(\mathrm{Cone}(u))[1]$$

This gives us a commutative diagram of extensions

$$\begin{tikzcd}
F(X) \arrow[r, "v_2"] \arrow[d, "="] & \mathrm{Cone}(-F(u)) \arrow[r, "w_2"] \arrow[d, "\cong"] & (FY)[1] \arrow[d, "((\alpha_Y^{-1})[1])^{-1}"] \\
F(X) \arrow[r, "F(w_3)"] \arrow[d, "="] & F(\mathrm{Cone}(u[-1])) \arrow[r, "F(v_3)"] \arrow[d, "\cong"] & F(Y[-1]) \arrow[d, "="] \\
F(X) \arrow[r, "F(w[-1])"] \arrow[d, "(\alpha_X^{-1})[1]"] & F(\mathrm{Cone}(u)[-1]) \arrow[r, "F(v[-1])"] \arrow[d, "(\alpha_{\mathrm{Cone}(u)}^{-1})[1]"] & F(Y[-1]) \arrow[d, "(\alpha_Y^{-1})[1]"] \\
F(X[1])[1] \arrow[r, "F(w)[1]"] & F(\mathrm{Cone}(u))[1] \arrow[r, "F(v)[1]"] & (FY)[1]
\end{tikzcd}$$

where $Y[-1] \xrightarrow{v_3} \mathrm{Cone}(u[-1]) \xrightarrow{w_3} X$ denote the extension corresponding to the cone of $u[-1]$. The result follows. □

We have the following result which tells us that a natural transformation between contravariant dg functor commute with the α^n.

Lemma 3.14. *Let $F : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(B)^{\mathrm{op}}$ and $G : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(B)^{\mathrm{op}}$ be dg functors. Also let $\psi \in Z^0(\mathcal{H}\mathrm{om}(F, G))$ be a natural transformation. Then the diagram*

$$\begin{tikzcd}
{[n] \circ F} \arrow[r, "{[n]\psi}"] \arrow[d, "\alpha^n"] & {[n] \circ G} \arrow[d, "\alpha^n"] \\
F \circ [-n] \arrow[r, "\psi_{[-n]}"] & G \circ [-n]
\end{tikzcd}$$

commute.

Proof. This is a straightforward calculation □

3.8 Triangulated categories

We start with a definition of a triangulated category. This version is taken from chapter 4 in [5] .

Definition 3.15. Let $\mathcal{C}$ be an additive category. We say that $\mathcal{C}$ is a triangulated category if it comes equipped with an additive automorphism

$$T : \mathcal{C} \to \mathcal{C}$$

called the *translation functor*, and a class of *distinguished triangles*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

also denoted (X, Y, Z, u, v, w), satisfying the following axioms

1. a) $X \xrightarrow{1_X} X \xrightarrow{0} 0 \xrightarrow{0} TX$ is a triangle
   b) Any triangle isomorphic to a distinguished triangle is distinguished
   c) Any morphism $X \xrightarrow{u} Y$ can be extended to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

2. A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

   is distinguished if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$$

   is distinguished.

3. Given a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted, "h"] & TX \arrow[d, "Tf"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX'
\end{tikzcd}$$

   where the top and bottom row are distinguished triangles, and the morphisms f and g makes the right square commute, then there exists a morphism $h : Z \to Z'$ such that the middle and the right square commutes.

4.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "1_X"] & Y \arrow[r, "v"] \arrow[d, "v"] & Z' \arrow[r, "w"] \arrow[d, dotted, "r"] & TX \arrow[d, "1_{TX}"] \\
X \arrow[r, "v \circ u"] & Z \arrow[r, "j"] \arrow[d, "h"] & Y' \arrow[r, "k"] \arrow[d, dotted, "s"] & TX \arrow[d, "Tu"] \\
& X' \arrow[r, "1_{X'}"] \arrow[d, "i"] & X' \arrow[r, "i"] \arrow[d, "Tv \circ i"] & TY \\
& TY \arrow[r, "Tv"] & TZ' &
\end{tikzcd}$$

Given distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z' \xrightarrow{w} TX$, $Y \xrightarrow{v} Z \xrightarrow{h} X' \xrightarrow{i} TY$, $X \xrightarrow{v \circ u} Z \xrightarrow{j} Y' \xrightarrow{k} TX$, there exists morphisms $r : Z' \to Y'$, $s : Y' \to X'$ such that the four squares in diagram commutes and

$$Z' \xrightarrow{r} Y' \xrightarrow{s} X' \xrightarrow{Tv \circ i} TZ'$$

is a distinguished triangle

Since T is an automorphism it has an inverse which we will denote by T^{-}. We have the following lemma for a triangulated category

Lemma 3.16. *Let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

be a distinguished triangle. Then $v \circ u = 0$.

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, dotted, "0"] & Y \arrow[r, "v"] \arrow[d, "v"] & Z \arrow[r, "w"] \arrow[d, "1_Z"] & TX \arrow[d, dotted, "0"] \\ 0 \arrow[r, "0"] & Z \arrow[r, "1_Z"] & Z \arrow[r, "0"] & 0 \end{tikzcd}$$

Note that the lower and the upper sequence are distinguished triangles. By shifting the triangles using axiom 2 and then applying axiom 3 we get that there must exists a morphism $X \to 0$ which makes the squares commutes. This morphism must necessarily be 0, so by considering the first square we get that $v \circ u = 0$ $\square$

By shifting triangles one easily sees that this implies that the composition of any two consecutive morphisms in a triangle is 0.

Lemma 3.17. *Let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

be a distinguished triangle and K an object of $\mathcal{C}$. We have long exact sequences

$$\ldots \xrightarrow{T^{-2}w\circ -} \mathrm{Hom}_{\mathcal{C}}(K, T^{-}X) \xrightarrow{T^{-}u\circ -} \mathrm{Hom}_{\mathcal{C}}(K, T^{-}Y) \xrightarrow{T^{-}v\circ -} \mathrm{Hom}_{\mathcal{C}}(K, T^{-}Z)$$
$$\xrightarrow{T^{-}w\circ -} \mathrm{Hom}_{\mathcal{C}}(K, X) \xrightarrow{u\circ -} \mathrm{Hom}_{\mathcal{C}}(K, Y) \xrightarrow{v\circ -} \mathrm{Hom}_{\mathcal{C}}(K, Z)$$
$$\xrightarrow{w\circ -} \mathrm{Hom}_{\mathcal{C}}(K, TX) \xrightarrow{Tu\circ -} \mathrm{Hom}_{\mathcal{C}}(K, TY) \xrightarrow{Tv\circ -} \mathrm{Hom}_{\mathcal{C}}(K, TZ) \xrightarrow{Tw\circ -} \ldots$$

and

$$\ldots \xrightarrow{-\circ Tw} \mathrm{Hom}_{\mathcal{C}}(TZ, K) \xrightarrow{-\circ Tv} \mathrm{Hom}_{\mathcal{C}}(TY, K) \xrightarrow{-\circ Tu} \mathrm{Hom}_{\mathcal{C}}(TX, K)$$
$$\xrightarrow{-\circ w} \mathrm{Hom}_{\mathcal{C}}(Z, K) \xrightarrow{-\circ v} \mathrm{Hom}_{\mathcal{C}}(Y, K) \xrightarrow{-\circ u} \mathrm{Hom}_{\mathcal{C}}(X, K)$$
$$\xrightarrow{-\circ T^{-}w} \mathrm{Hom}_{\mathcal{C}}(T^{-}Z, K) \xrightarrow{-\circ T^{-}v} \mathrm{Hom}_{\mathcal{C}}(T^{-}Y, K) \xrightarrow{-\circ T^{-}u} \mathrm{Hom}_{\mathcal{C}}(T^{-}X, K) \xrightarrow{T^{-2}w} \ldots$$

Proof. By the symmetry property of distinguished triangles we only need to check that

$$\mathrm{Hom}_{\mathcal{C}}(K,X) \xrightarrow{u\circ -} \mathrm{Hom}_{\mathcal{C}}(K,Y) \xrightarrow{v\circ -} \mathrm{Hom}_{\mathcal{C}}(K,Z) \tag{3.2}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(Z,K) \xrightarrow{-\circ v} \mathrm{Hom}_{\mathcal{C}}(Y,K) \xrightarrow{-\circ u} \mathrm{Hom}_{\mathcal{C}}(X,K) \tag{3.3}$$

are exact. We observe by lemma 3.16 that the composition of the two maps are 0 in both sequences. For the exactness of (3.2) consider $h \in \mathrm{Hom}_{\mathcal{C}}(K,Y)$ satisfying $v \circ h = 0$. We then have the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & TX \\
K \arrow[u, dotted, "k"] \arrow[r, "1_K"] & K \arrow[u, "h"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & K \arrow[u, dotted, "Tk"]
\end{tikzcd}$$

where the lower and upper sequence are distinguished triangles and the middle square commutes and the left square commutes. By axiom 3 we have that there exists a morphism $k : K \to X$ satisfying $u \circ k = h$, so (3.2) is exact. For (3.3) assume $l \in \mathrm{Hom}_{\mathcal{C}}(Y,K)$ satisfies $l \circ u = 0$. We then have a digram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "0"] & Y \arrow[r, "v"] \arrow[d, "l"] & Z \arrow[r, "w"] \arrow[d, dotted, "m"] & TX \arrow[d, "0"] \\
0 \arrow[r, "0"] & K \arrow[r, "1_K"] & K \arrow[r, "0"] & 0
\end{tikzcd}$$

where the upper and lower sequence are distinguished triangles. By axiom 3 we have that there exists $m : Z \to K$ satisfying $m \circ v = l$, and hence the sequence (3.3) is also exact $\square$

This gives us the following corollary

Corollary 3.18. *Consider the commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[r, "u_1"] \arrow[d, "f"] & Y_1 \arrow[r, "v_1"] \arrow[d, "g"] & Z_1 \arrow[r, "w_1"] \arrow[d, "h"] & TX_1 \arrow[d, "Tf"] \\
X_2 \arrow[r, "u_2"] & Y_2 \arrow[r, "v_2"] & Z_2 \arrow[r, "w_2"] & TX_2
\end{tikzcd}$$

where the upper and lower rows are distinguished triangles. If two out of f, g and h are isomorphisms, then the third one also is.

Proof. By axiom 2 we see that it is enough to show that h is an isomorphism when f and g are isomorphisms. Applying $\mathrm{Hom}(Z_2, -)$ we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(Z_2, X_1) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(Z_2, Y_1) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(Z_2, Z_1) \arrow[r] \arrow[d, "h \circ -"] & \mathrm{Hom}(Z_2, TX_1) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(Z_2, TY_1) \arrow[d, "\cong"] \\
\mathrm{Hom}(Z_2, X_2) \arrow[r] & \mathrm{Hom}(Z_2, Y_2) \arrow[r] & \mathrm{Hom}(Z_2, Z_2) \arrow[r] & \mathrm{Hom}(Z_2, TX_2) \arrow[r] & \mathrm{Hom}(Z_2, TY_2)
\end{tikzcd}$$

where the upper and lower rows are exact sequences. By the five lemma we get that

$$h \circ - : \mathrm{Hom}(Z_2, Z_1) \to \mathrm{Hom}(Z_2, Z_2)$$

is an isomorphism, and hence h has a left inverse. Similarly by applying $\mathrm{Hom}(-, Z_1)$ to the diagram we can show that h has a right inverse. $\square$

We also need to say what a functor between triangulated categories should be

Definition 3.19. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. An additive functor

$$F : \mathcal{C} \to \mathcal{D}$$

is called a *triangle functor* if there exists a natural isomorphism

$$F \circ T \xrightarrow{\eta} T \circ F$$

and if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

is a distinguished triangle in $\mathcal{C}$, then

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\eta_X \circ F(w)} TF(X)$$

is a distinguished triangle in $\mathcal{D}$.

3.9 Homotopy categories

Let A be a dg algebra. Recall that we have the homotopy category $\mathcal{H}(A)$ defined in section 3.2. The objects in $\mathcal{H}(A)$ are the same as the objects in $C(A)$, and the morphisms are morphisms in $C(A)$ modulo nullhomotopic maps. The shift functor $[1] : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(A)$ is a dg functor and therefore induces a well defined additive automorphism

$$[1] : \mathcal{H}(A) \to \mathcal{H}(A)$$

taking an object M to $M[1]$ and a morphism f to $[1](f) = f[1]$. We call a sequence of map in $\mathcal{H}(A)$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

for a *triangle*, and denoted it by (u, v, w). A morphism of two triangles (u_1, v_1, w_1) and (u_2, v_2, w_2) is a commutative diagram of the form

$$\begin{tikzcd}
X_1 \arrow[r, "u_1"] \arrow[d, "f"] & Y_1 \arrow[r, "v_1"] \arrow[d, "g"] & Z_1 \arrow[r, "w_1"] \arrow[d, "h"] & X_1[1] \arrow[d, "f[1]"] \\
X_2 \arrow[r, "u_2"] & Y_2 \arrow[r, "v_2"] & Z_2 \arrow[r, "w_2"] & X_2[1]
\end{tikzcd}$$

A triangle of the form

$$X \xrightarrow{u} Y \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} X[1]$$

is called a *strict triangle*. We say that a triangle is *exact* if it is isomorphic to a strict triangle. The proof of the following theorem is similar to the case when A is a ring

Theorem 3.20. *Let A be a dg algebra. The category $\mathcal{H}(A)$ is triangulated with*

$$[1] : \mathcal{H}(A) \to \mathcal{H}(A)$$

as translation functor, and where the distinguished triangles are the exact triangles.

Let K be a dg A-module. We say that K is *homotopically projective* if for any acyclic dg A-module N we have that

$$\mathrm{Hom}_{\mathcal{H}(A)}(K, N) = 0$$

We say that K is *homotopically injective* if for any acyclic dg A-module N we have that

$$\mathrm{Hom}_{\mathcal{H}(A)}(N, K) = 0$$

The homotopically projective modules plays a similar role in $\mathcal{H}(A)$ as bounded above chain complexes with projective components does in $\mathcal{H}^-(B)$ when B is a ring. Let $\mathcal{H}_p(A)$ (resp $\mathcal{H}_i(A)$) denote the full subcategory of $\mathcal{H}(A)$ with object homotopically projective (homotopically injective) dg A-modules.

Lemma 3.21. *The category $\mathcal{H}_p(A)$ (resp $\mathcal{H}_i(A)$) is triangulated and closed under infinite direct sums and direct summands.*

Proof. Let N be an acyclic dg A-modules. We first show that $\mathcal{H}_p(A)$ is closed under direct sums and direct summands. Let $(K_i)_{i\in\mathcal{I}}$ be a collection of dg A-modules. Then

$$\mathrm{Hom}_{\mathcal{H}(A)}(\bigoplus_{i\in\mathcal{I}} K_i, N) \cong \prod_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{H}(A)}(K_i, N)$$

and hence $\mathrm{Hom}_{\mathcal{H}(A)}(\bigoplus_{i\in\mathcal{I}} K_i, N) = 0$ iff $\mathrm{Hom}_{\mathcal{H}(A)}(K_i, N) = 0$ for all i. Therefore $\bigoplus_{i\in\mathcal{I}} K_i$ is homotopically projective iff K_i is homotopically projective for all i. Now assume K is a homotopically projective module. We have an isomorphism

$$\mathcal{H}\mathrm{om}_A(K[1], N) \cong \mathcal{H}\mathrm{om}_A(K, N[-1])$$

by lemma 3.5. Taking homology in degree 0 we get that

$$\mathrm{Hom}_{\mathcal{H}(A)}(K[1], N) \cong \mathrm{Hom}_{\mathcal{H}(A)}(K, N[-1]) \cong 0$$

since acyclic modules are closed under the shift functor. This implies that $K[1]$ is also homotopically projective. It remains to show that if

$$X \to K \to Y \to X[1]$$

is a strict triangle with X and Y homotopically projective, then K must be homotopically projective. Since $\mathcal{H}(A)$ is triangulated we have an exact sequence

$$\mathrm{Hom}_{\mathcal{H}(A)}(Y, N) \to \mathrm{Hom}_{\mathcal{H}(A)}(K, N) \to \mathrm{Hom}_{\mathcal{H}(A)}(X, N)$$

by lemma 3.17. Since $\mathrm{Hom}_{\mathcal{H}(A)}(Z, N) = 0$ and $\mathrm{Hom}_{\mathcal{H}(A)}(X, N) = 0$ we get that $\mathrm{Hom}_{\mathcal{H}(A)}(K, N) = 0$ and hence K is homotopically projective. □

Recall that we defined strictly perfect modules in 3.9

Lemma 3.22. *Let M be a strictly perfect dg A-module. Then M is homotopically projective.*

Proof. This follows from $\mathcal{H}_p(A)$ being triangulated, containing A, and closed under direct summands □

A proof of the next theorem can be found in the appendix of [10].

Theorem 3.23. *Let M be a dg A-module*

- *There exists a quasi-isomorphism*

  $$pX \to X$$

  with pX homotopically projective. This assignment induces a triangle functor

  $$p : \mathcal{H}(A) \to \mathcal{H}(A)$$

  which makes the quasi-isomorphisms $pX \to X$ into a natural transformation between p and the identity functor.

- *There exists a quasi-isomorphism*

$$X \to iX$$

with iX homotopically injective. This assignment induces a triangle functor

$$i : \mathcal{H}(A) \to \mathcal{H}_i(A)$$

which makes the quasi-isomorphism $X \to iX$ into a natural transformation between the identity functor and i.

3.10 Derived categories

Let A be a dg algebra, and consider the collection S of all quasi-isomorphism in $\mathcal{H}(A)$. If we localise the category $\mathcal{H}(A)$ with respect to S we get the the *derived categories* of A, denoted by $D(A)$. It shares a lot of the same properties as the derived category of an ordinary algebra. In particular a morphism $X \to Y$ in $D(A)$ can be represented as

$$X \xleftarrow{s} Z \xrightarrow{f} Y$$

and as

$$X \xrightarrow{f'} Z' \xleftarrow{s'} Y$$

where f, f' are morphisms and s,s' are quasi isomorphisms in $\mathcal{H}(A)$. The shift functor $[1] : \mathcal{H}(A) \to \mathcal{H}(A)$ preserves quasi-isomorphism and therefore induces a well defined functor

$$[1] : D(A) \to D(A)$$

which sends a morphism

$$X \xleftarrow{s} Z \xrightarrow{f} Y$$

to

$$X[1] \xleftarrow{s[1]} Z[1] \xrightarrow{f[1]} Y[1]$$

We have the following theorem

Theorem 3.24. *Let A be a dg algebra. The category $D(A)$ is triangulated with*

$$[1] : D(A) \to D(A)$$

as the translation functor. The distinguished triangles are the ones which are isomorphic in $D(A)$ to an exact triangle.

Recall that we have a functor $p : \mathcal{H}(A) \to \mathcal{H}_p(A)$ from section 3.9. These are very useful when studying the derived category.

Theorem 3.25. *Let A be a dg algebra.*

- *The functor $p : \mathcal{H}(A) \to \mathcal{H}(A)$ induces a triangle functor*

$$p : D(A) \to \mathcal{H}(A)$$

  which is fully faithful and left adjoint to the projection functor $\mathcal{H}(A) \to D(A)$.

- *The functor $i : \mathcal{H}(A) \to \mathcal{H}(A)$ induces a triangle functor*

$$i : D(A) \to \mathcal{H}(A)$$

  which is fully faithful and right adjoint to the projection functor $\mathcal{H}(A) \to D(A)$.

Below we define some subcategories of $D(A)$ which we will need later.

Definition 3.26. Let $\mathrm{perf}(A)$ be the smallest full triangulated subcategory of $D(A)$ closed under direct summands. A dg A-module M is called *perfect* if it is an object of $\mathrm{perf}(A)$.

Note that a chain complex over an ordinary algebra is perfect if and only if it is quasi-isomorphic to a bounded complex with projective components.

Definition 3.27. Let k be a field and A a dg k-algebra. We then have a full triangulated subcategory of $D(A)$ formed by dg A-modules M such that

$$\sum \dim_k H^p(M)$$

is finite dimensional. We will denote this by $D_{fd}(A)$.

If A is a finite dimensional algebra then $D_{fd}(A) = D^b(\mathrm{mod}\, A)$.

4. PREDUALITY FUNCTORS

In this chapter we will consider dg algebras B equipped with an *involution*

$$\tau : B \to B^{\mathrm{op}}$$

which is a morphism of dg algebras satisfying $\tau^2 = 1_B$.

Definition 4.1. A preduality functor on a category $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ equipped with a natural transformation

$$\phi : 1_{\mathcal{C}} \to VV$$

satisfying $V\phi \circ \phi_V = 1_V$.

We will show that a dg algebra B with an involution has a dg preduality functor

$$V : \mathcal{C}_{dg}(B) \to \mathcal{C}_{dg}(B)^{\mathrm{op}}$$

We will also show that $[n] \circ V$ is a preduality functor if V is a preduality functor, where $[n]$ is the shift functor. This will be necessary for the definition of the deformed n-Calabi-Yau completion of a homologically smooth algebra in the next section. We will also investigate the situation where

$$F : A \to B$$

is a morphism of dg algebras satisfying $F \circ \tau_A = \tau_B \circ F$ where τ_A and τ_B are involutions on A and B.

4.1 Extending involutions to preduality functors

Let A be a dg algebra with an involution τ. Assume that M is a left dg A-module. We then have a right dg A-module $\overline{M}$ with module structure given by

$$m * a = (-1)^{|m|\cdot|a|}\tau(a) \cdot m$$

This induces a dg functor

$$\tau' : C_{dg}(A^{\mathrm{op}}) \to C_{dg}(A)$$

Now let N be a right dg A-module. We have a dg functor * sending N to $N^* = \mathcal{H}\mathrm{om}_A(N, A)$ and a morphism $f : N_1 \to N_2$ to

$$f^* = \mathcal{H}\mathrm{om}_A(f, A) : \mathcal{H}\mathrm{om}_A(N_2, A) \to \mathcal{H}\mathrm{om}_A(N_1, A)$$

given by $f^*(g) = (-1)^{|f|\cdot|g|} g \circ f$. We let

$$V = \tau' \circ^* \tag{4.1}$$

be the composition, so V is a functor

$$V : C_{dg}(A) \to C_{dg}(A)^{\mathrm{op}}$$

It takes a right module N to $\overline{\mathcal{H}\mathrm{om}_A(N, A)}$. We also have a natural maps

$$\phi_N : N \to VV(N)$$

given by $\phi_N(n)(f) = (-1)^{|f|\cdot|n|}\tau(f(n))$.

Proposition 4.2. *We use the same notation as above.*

1. *V is a differential graded functor*
2. *$\phi_N \in Z^0(\mathcal{H}\mathrm{om}_A(N, VVN)) = \mathrm{Hom}_A(N, VVN)$*
3. *V is a preduality functor*

Proof. For the first part observe that V is the composition of the dg functors τ' and $\mathcal{H}\mathrm{om}_A(-, A)$, and is therefore a dg functor.
Let M be a right dg A-module. Since

$$\begin{aligned}
&\phi_N(n \cdot a)(f) = (-1)^{|f|\cdot(|n|+|a|)} \cdot \tau(f(n \cdot a)) = (-1)^{|f|\cdot(|n|+|a|)} \cdot \tau(f(n) \cdot a) \\
&= (-1)^{|f|\cdot|n|+|a|\cdot|n|} \cdot \tau(a) \cdot \tau(f(n)) = (-1)^{|a|\cdot|n|} \cdot \tau(a) \cdot (\phi_N(n)(f)) \\
&= (\phi_N(n) * a)(f)
\end{aligned}$$

we get that

$$\phi_N(n \cdot a) = \phi_N(n) * a$$

It therefore only remain to show that ϕ_N commutes with the differential in order to prove the second part. This follows since

$$\phi_N(d(n))(f) = (-1)^{|n|\cdot|f|+|f|} \cdot f(d(n))$$

and

$$\begin{aligned}
&d(\phi_N(n))(f) = d \circ \phi_N(n)(f) - (-1)^{|n|} \cdot \phi_N(n) \circ d(f) \\
&= (-1)^{|n|\cdot|f|} \cdot d(f(n)) - (-1)^{|n|\cdot|f|} \cdot d(f)(n) \\
&= (-1)^{|n|\cdot|f|} \cdot d(f(n)) - (-1)^{|n|\cdot|f|} \cdot d(f(n)) + (-1)^{|n|\cdot|f|+|f|} \cdot f(d(n)) \\
&= (-1)^{|n|\cdot|f|+|f|} \cdot f(d(n))
\end{aligned}$$

For the last part we want to show

$$VM \xrightarrow{\phi_{VM}} VVVM \xrightarrow{V\phi_M} VM$$

is equal to 1_{VM}. So let $f \in VM = \overline{\mathcal{H}\mathrm{om}_A(M,A)}$ and $m \in M$. Then

$$\begin{aligned} &V\phi_M(\phi_{VM}(f))(m) = \phi_{VM}(f)(\phi_M(m)) = (-1)^{|m|\cdot|f|}\tau(\phi_M(m)(f)) \\ &= \tau(\tau(f(m))) = f(m) \end{aligned}$$

and the result follows. □

4.2 Properties of dg preduality functors

Let F be a functor satisfying proposition 4.2, i.e F is a dg functor

$$F : \mathcal{C}_{dg}(A) \to \mathcal{C}_{dg}(A)^{\mathrm{op}}$$

and we have morphisms $\phi_M : M \to FFM$ natural in M which satisfy

$$F(\phi_M) \circ \phi_{FM} = 1_{FM}$$

Let F_n denote the composition $[n] \circ F$. We then have a natural transformation

$$1 \xrightarrow{\phi} FF \xrightarrow{=} F[-n][n]F \xrightarrow{(\alpha^n_{[n]F})^{-1}} [n]F[n]F = F_n \circ F_n \tag{4.2}$$

which we will denote by ϕ^n.

Proposition 4.3. *$F_n = [n] \circ F$ together with ϕ^n is a preduality dg functor satisfying proposition 4.2*

Proof. It is obvious that F_n satisfy property 1 and 2 of proposition 4.2. We therefore only need to show that

$$F_n \xrightarrow{\phi^n_{F_n}} F_nF_nF_n \xrightarrow{F_n(\phi^n)} F_n$$

is the identity. This composition is the same as

$$\begin{aligned} [n]F \xrightarrow{\phi_{[n]F}} FF[n]F \xrightarrow{=} F[-n][n]F[n]F \xrightarrow{(\alpha^n_{[n]F[n]F})^{-1}} [n]F[n]F[n]F \\ \xrightarrow{[n]F((\alpha^n_{[n]F})^{-1})} [n]FF[-n][n]F \xrightarrow{=} [n]FFF \xrightarrow{[n]F(\phi)} [n]F \end{aligned}$$

We will denote this composition by Φ. We first need to show that

$$\begin{tikzcd} {[n]F} \arrow[r, "\phi_{[n]F}"] \arrow[ddr, "{[n](\phi_F)}"'] & FF[n]F \arrow[d, "F(\alpha^{-n}_F)"] \\ & F[-n]FF \arrow[d, "(\alpha^n)^{-1}_{FF}"] \\ & {[n]FFF} \end{tikzcd} \tag{4.3}$$

commutes. Note that

$$F(\alpha_F^{-n}) = (-1)^{n(n+1)/2} \cdot F(\Sigma^n_{[-n]FF}) \circ FF(\Sigma^{-n}_{[n]F})$$

and

$$(\alpha^n)^{-1}_{FF} = (-1)^{n(n-1)/2} \cdot \Sigma^n_{FFF} \circ F(\Sigma^{-n}_{FF})$$

so

$$(\alpha^n)^{-1}_{FF} \circ F(\alpha_F^{-n}) = \Sigma^n_{FFF} \circ FF(\Sigma^{-n}_{[n]F})$$

Also note that the square

$$\begin{tikzcd}
{[n]F} \arrow[r, "\phi_{[n]F}"] \arrow[d, "\Sigma^{-n}_{[n]F}"'] & FF[n]F \arrow[d, "FF(\Sigma^{-n}_{[n]F})"] \\
F \arrow[r, "\phi_F"] & FFF
\end{tikzcd}$$

commutes by naturality of ϕ. Hence

$$\begin{aligned}
(\alpha^n)^{-1}_{FF} \circ F(\alpha_F^{-n}) \circ \phi_{[n]F} &= \Sigma^n_{FFF} \circ FF(\Sigma^{-n}_{[n]F}) \circ \phi_{[n]F} \\
&= \Sigma^n_{FFF} \circ \phi_F \circ \Sigma^{-n}_{[n]F} \\
&= [n](\phi_F)
\end{aligned}$$

and 4.3 commutes. Therefore Φ can be written as

$$[n]F(\phi) \circ \beta \circ [n](\phi_F)$$

Where $\beta : [n]FFF \to [n]FFF$ is given by

$$\beta = [n]F((\alpha^n_{[n]F})^{-1}) \circ (\alpha^n_{[n]F[n]F})^{-1} \circ F(\alpha_F^{-n})^{-1} \circ \alpha^n_{FF}$$

We also have that

$$[n]F(\phi) \circ [n](\phi_F) = [n](F(\phi) \circ \phi_F) = 1_{[n]F}$$

since F is a preduality functor. We therefore only need to show that $\beta = 1$.

Consider the diagram

$$
\begin{tikzcd}
{[n]FFF} \arrow[r, dash, "="] \arrow[dr, dash, "="] \arrow[dd, dash, "\cong"] & {[n]FF[-n][n]F} \arrow[r, dash, "\cong"] & {[n]F[n]F[n]F} \arrow[dl, dash, "\cong"'] \arrow[ddd, dash, "="] \\
 & {[n]F[n][-n]FF} \arrow[d, dash, "\cong"] & \\
F[-n]FF \arrow[r, dash, "="] \arrow[d, dash, "\cong"] & F[-n][n][-n]FF \arrow[d, dash, "\cong"] & \\
FF[n]F \arrow[r, dash, "="] & F[-n][n]F[n]F \arrow[r, dash, "\cong"] & {[n]F[n]F[n]F}
\end{tikzcd}
$$

where the maps are the natural ones. Note that we get β if we follow the isomorphisms on the boundary of the square counterclockwise around the diagram starting in the upper left corner. Also note that the two left diagrams and the right diagram commutes since the maps are natural isomorphisms, while the upper diagram commutes because of the commutativity of diagram (3.1) in proposition 3.12. The commutativity of the small diagrams implies that the big diagram commutes, and hence $\beta = 1$. Therefore F_n is a preduality functor.

□

Let $f : X \to FX$ be a closed morphism. We say that f is (F, ϕ)-*symmetric* if

$$f = F(f) \circ \phi_X$$

and (F, ϕ)-*antisymmetric* if

$$f = -F(f) \circ \phi_X$$

Proposition 4.4. (2.5 in [7]) *Let $f : X \to FX$ be a closed morphism.*

- *If f is an (F, ϕ)-antisymmetric morphism then the cone of f has a $([1] \circ F, \phi^1)$-symmetric morphism*

  $$g : \mathrm{Cone}(f) \to [1]F\mathrm{Cone}(f)$$

  Furthermore, if ϕ_X is a quasi-isomorphism then g is a quasi-isomorphism

- *If f is an (F, ϕ)-symmetric morphism then the cone of f has a $([1] \circ F, \phi^1)$-antisymmetric morphism*

  $$h : \mathrm{Cone}(f) \to [1]F\mathrm{Cone}(f)$$

  Furthermore, if ϕ_X is a quasi-isomorphism then g is a quasi-isomorphism

Proof. We will only show the second part. The first part is similar. So assume we have a symmetric closed morphism $f : X \to FX$. We then have a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"] & FX \arrow[d, "-1"] \\ FFX \arrow[r, "-F(f)"] & FX \end{tikzcd}$$

From lemma 3.4 we get a closed morphism

$$\begin{pmatrix} -1 & 0 \\ 0 & \phi_X[1] \end{pmatrix} : \mathrm{Cone}(f) \to \mathrm{Cone}(-F(f))$$

with the identification $\mathrm{Cone}(f) = FX \oplus X[1]$ and $\mathrm{Cone}(-F(f)) = FX \oplus (FFX)[1]$ as graded modules. Also by lemma 3.13 we have an isomorphism

$$\begin{pmatrix} (\alpha_X^{-1})[1] & 0 \\ 0 & 1 \end{pmatrix} : \mathrm{Cone}(-F(f)) \to F_1(\mathrm{Cone}(f))$$

where $F_1(\mathrm{Cone}(f)) = F_1(X[1])[1] \oplus F_1FX$ as a graded module. We let

$$g = \begin{pmatrix} -(\alpha_X^{-1})[1] & 0 \\ 0 & \phi_X[1] \end{pmatrix} : \mathrm{Cone}(f) \to F_1(\mathrm{Cone}(f))$$

denote the composition of these maps. Then g gives us a commutative diagram

$$\begin{tikzcd} FX \arrow[r, "v_1"] \arrow[d, "-(\alpha_X^{-1})[1]"] & \mathrm{Cone}(f) \arrow[r, "w_1"] \arrow[d, "g"] & X[1] \arrow[d, "\phi_X[1]"] \\ F_1(X[1]) \arrow[r, "F_1(w_1)"] & F_1\mathrm{Cone}(f) \arrow[r, "F_1(v_1)"] & F_1FX \end{tikzcd} \tag{4.4}$$

Here v_1, w_1 are the inclusion and projection induced by the cone. We also have isomorphisms

$$((\alpha_X^{-1})[1])^{-1} : F_1(X[1]) \to FX$$

and

$$F_1((\alpha_X^{-1})[1])^{-1} : F_1FX \to F_1F_1(X[1])$$

So we can write $\mathrm{Cone}(f) = F_1(X[1]) \oplus X[1]$ and $F_1\mathrm{Cone}(f) = F_1(X[1]) \oplus F_1F_1(X[1])$. With this identification we get

$$g = \begin{pmatrix} -1 & 0 \\ 0 & F_1((\alpha_X^{-1})[1])^{-1} \circ \phi_X[1] \end{pmatrix} : \mathrm{Cone}(f) \to F_1(\mathrm{Cone}(f))$$

Also it is not hard to see that

$$\phi^1_{X[1]} = F_1((\alpha_X^{-1})[1])^{-1} \circ \phi_X[1]$$

Thus

$$g = \begin{pmatrix} -1 & 0 \\ 0 & \phi^1_{X[1]} \end{pmatrix} : \mathrm{Cone}(f) \to F_1(\mathrm{Cone}(f))$$

and we have a commutative diagram

$$\begin{tikzcd} F_1(X[1]) \arrow[r, "v"] \arrow[d, "-1"] & \mathrm{Cone}(f) \arrow[r, "w"] \arrow[d, "g"] & X[1] \arrow[d, "\phi^1_{X[1]}"] \\ F_1(X[1]) \arrow[r, "F_1(w)"] & F_1\mathrm{Cone}(f) \arrow[r, "F_1(v)"] & F_1F_1(X[1]) \end{tikzcd} \tag{4.5}$$

where $v = v_1 \circ ((\alpha_X^{-1})[1])^{-1}$ and $w = w_1$. Note that

$$\phi^1_{\mathrm{Cone}(f)} = \begin{pmatrix} \phi^1_{F_1(X[1])} & 0 \\ 0 & \phi^1_{X[1]} \end{pmatrix} : \mathrm{Cone}(f) \to F_1F_1\mathrm{Cone}(f)$$

and

$$F_1(g) = \begin{pmatrix} F_1(\phi^1_{X[1]}) & 0 \\ 0 & -1 \end{pmatrix} : F_1F_1\mathrm{Cone}(f) \to F_1\mathrm{Cone}(f)$$

where $F_1F_1\mathrm{Cone}(f) = F_1F_1F_1(X[1]) \oplus F_1F_1(X[1])$ as a graded module. Therefore

$$g = -F_1(g) \circ \phi^1_{\mathrm{Cone}(f)}$$

Now assume ϕ_X is a quasi-isomorphism. Consider the commutative diagram (4.5). We have that the lower and upper sequence is contained in a distinguished triangle by lemma 3.8. Hence g is a quasi-iso by corollary 3.18. □

We will also need the following result for later

Lemma 4.5. *Assume*

$$\phi_A : A \to FFA$$

is an isomorphism. If X is a strictly perfect dg A-module then

$$\phi_X : X \to FFX$$

is an isomorphism.

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{C}_{dg}(A)$ with objects X such that ϕ_X is an isomorphism. First observe that A is an object of $\mathcal{C}$. Also since the diagram

$$\begin{tikzcd}
[1](X) \arrow[r, "\phi_{[1](X)}"] \arrow[dr, "\phi_X[1]"'] & FF[1](X) \arrow[d, "\cong"] \\
 & (FFX)
\end{tikzcd}$$

commutes we get that if $X[1] \in \mathcal{C}$ if $X \in \mathcal{C}$. Now assume that

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

is an extension and $X_1 \in \mathcal{C}$ and $X_3 \in \mathcal{C}$. We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "\phi_{X_1}"] & X_2 \arrow[r] \arrow[d, "\phi_{X_2}"] & X_3 \arrow[r] \arrow[d, "\phi_{X_2}"] & 0 \\
0 \arrow[r] & FFX_1 \arrow[r] & FFX_2 \arrow[r] & FFX_3 \arrow[r] & 0
\end{tikzcd}$$

where the upper and lower row are exact in $C(A)$. By the five-lemma we get that ϕ_{X_2} is an isomorphism, and hence $X_2 \in \mathcal{C}$. Therefore $\mathcal{C}$ is closed under extensions. By naturality of ϕ it is easy to see that if $X \in \mathcal{C}$ and Y is a direct summand of X then $Y \in \mathcal{C}$. Hence $\mathcal{C}$ contains strictperf(A), and the result follows. $\square$

4.3 Derived preduality functors

Let A be a dg algebra with an involution τ_A. Let V denote the extension of the preduality functor given in (4.1), and let $V_n = [n] \circ V$. Since V_n is a dg functor it induces a well defined triangle functor on the homotopy categories also denoted by

$$V_n : \mathcal{H}(A) \to \mathcal{H}(A)^{\mathrm{op}}$$

We define the *total derived functor* of V_n to be

$$DV_n = \pi \circ V_n \circ p$$

where $p : D(A) \to \mathcal{H}(A)$ is the homotopically projective resolution functor in Theorem 3.25 and $\pi : \mathcal{H}(A) \to D(A)$ is the projection functor. Note that

$$DV_n : D(A) \to D(A)^{\mathrm{op}}$$

is a triangle functor. Consider the natural morphism

$$\phi^n : 1 \to V_n \circ V_n$$

We want to define a corresponding natural transformation ϕ'^n in $D(A)$ for the derived functors DV_n. If we let

$$q : p \to 1$$

denote the quasi-isomorphism in Theorem 3.23, then ϕ'^n is the composition

$$1 \xrightarrow{q^{-1}} p \xrightarrow{\phi^n_p} V_n \circ V_n \circ p \xrightarrow{V_n(q_{V_n \circ p})} V_n \circ p \circ V_n \circ p = DV_n \circ DV_n$$

Lemma 4.6. *If $f : M \to N$ is a morphism between homotopically projective dg A-modules. Then*

$$DV_n(f) = V_n(f)$$

Also if X, $V_n(X)$ are homotopically projective then

$$\phi'^n_X = \phi^n_X$$

Proof. This follows from the fact that p acts as the identity functor on homotopically projective modules. □

4.4 Relations with hom and tensor product

Let A and B be dg algebras with involutions τ_A and τ_B. Furthermore let $G : A \to B$ be a morphism of dg algebras satisfying

$$G \circ \tau_A = \tau_B \circ G$$

We want to investigate how the preduality functors on $\mathcal{C}_{dg}(A)$ and $\mathcal{C}_{dg}(B)$ relate.

Lemma 4.7. *Let M and N be right dg A modules. We then have a natural morphism of chain complexes*

$$N \otimes_A \mathcal{H}\mathrm{om}_A(M, A) \xrightarrow{\psi} \mathcal{H}\mathrm{om}_A(M, N)$$

given by

$$\psi(n \otimes f)(m) = n \cdot f(m)$$

Proof. We have a natural isomorphism

$$N \cong \mathcal{H}\mathrm{om}_A(A, N)$$

given by sending $n \in N$ to $g_n \in \mathcal{H}\mathrm{om}_A(A, N)$ defined by

$$g_n(a) = n \cdot a$$

With this identification we see that ψ is just the composition map, and hence the result follows. □

We will denote by $*$ the induced module structure on $\overline{M}$ when M is a dg module.

Lemma 4.8. *Let M be a right dg A modules and N a left dg A module. There is a natural isomorphism*

$$M \otimes_A N \cong \overline{N} \otimes_A \overline{M}$$

given by sending $m \otimes n$ to $(-1)^{|m|\cdot|n|} \cdot n \otimes m$.
Furthermore, if M has a left dg B-module structure (resp N has a right dg B-module structure) then the map above induces an isomorphism

$$\overline{M \otimes_A N} \cong \overline{N} \otimes_A \overline{M}$$

of right B-modules (resp left B-modules).

Proof. Let the map be denoted by η. We want to check that η is well defined. We have that

$$\eta(m \cdot a \otimes n) = (-1)^{(|m|+|a|)\cdot|n|} \cdot n \otimes (m \cdot a) = (-1)^{(|m|+|a|)\cdot|n|+|a|\cdot|m|} \cdot n \otimes \tau(a) * m$$

and

$$\eta(m \otimes a \cdot n) = (-1)^{(|n|+|a|)\cdot|m|} \cdot (a \cdot n) \otimes m = (-1)^{(|n|+|a|)\cdot|m|+|a|\cdot|n|} \cdot n * \tau(a) \otimes m$$

and hence is well defined. It is easy to see that η is a chain map and an isomorphism in both cases, so the result follows □

Note that the morphism $G : A \to B$ makes B into a dg A-module.

Lemma 4.9. *Let M and N be right dg A modules. We have a natural isomorphism*

$$\mathcal{H}\mathrm{om}_A(M, N) \cong \mathcal{H}\mathrm{om}_B(M \otimes_A B, N)$$

given by sending $f \in \mathcal{H}\mathrm{om}_A(M, N)$ to the map

$$m \otimes b \to f(m) \cdot b$$

Proof. This is a straightforward calculation □

The relation $G \circ \tau_A = \tau_B \circ G$ gives us the following result

Lemma 4.10. *There is an isomorphism of bi A-modules*

$$\overline{B} \cong B$$

given by sending b to $\tau_B(b)$.

Proof. We denote both the A-module structure of B and the multiplication in B by $\cdot$, so

$$a_1 \cdot b \cdot a_2 = G(a_1) \cdot b \cdot G(a_2).$$

where $a_1 \in A$, $a_2 \in A$ and $b \in B$. We have that

$$\begin{aligned}\tau_B(a_1 * b * a_2) &= (-1)^{|a_1|\cdot|b|+|a_1|\cdot|a_2|+|b|\cdot|a_2|} \cdot \tau_B(G(\tau_A(a_2)) \cdot b \cdot G(\tau_A(a_1))) \\ &= (-1)^{|a_1|\cdot|b|+|a_1|\cdot|a_2|+|b|\cdot|a_2|} \cdot \tau_B(\tau_B(G(a_2)) \cdot b \cdot \tau_B(G(a_1))) \\ &= G(a_1) \cdot \tau_B(b) \cdot G(a_2) = a_1 \cdot \tau_B(b) \cdot a_2\end{aligned}$$

and hence τ_B is a morphism of A-modules. Since it is obviously an isomorphism we are done. □

Now consider the tensor product

$$V(M) \otimes_A B = \overline{\mathcal{H}\mathrm{om}_A(M, A)} \otimes_A B$$

where M is a right dg A-module. From the above lemmas we get a sequence of maps

$$\begin{aligned}\overline{\mathcal{H}\mathrm{om}_A(M, A)} \otimes_A B \cong \overline{\overline{B} \otimes_A \mathcal{H}\mathrm{om}_A(M, A)} &\to \overline{\mathcal{H}\mathrm{om}_A(M, \overline{B})} \\ \cong \overline{\mathcal{H}\mathrm{om}_A(M, B)} &\cong \overline{\mathcal{H}\mathrm{om}_B(M \otimes_A B, B)}\end{aligned}$$

This gives us the following result

Lemma 4.11. (2.8 in [7]) *We have a natural morphism of right dg B-modules*

$$V(M) \otimes_A B \to V(M \otimes_A B)$$

given by sending $f \otimes b_1$ to the map

$$m \otimes b_2 \to (-1)^{|b_1|\cdot|f|} \cdot \tau_B(b_1) \cdot f(m) \cdot b_2$$

Furthermore, if M is strictly perfect then the map is an isomorphism.

Proof. Denote the map by ν_M. The first part follows from our discussion above. For the second part let $\mathcal{C}$ denote the full subcategory of $\mathcal{C}_{dg}(A)$ consisting of objects M such that ν_M is an isomorphism. We want to show that $\mathcal{C}$ contains strictperf(A). Observe first that ν is a morphism between contraviariant dg functors, i.e

$$\nu \in Z^0(\mathcal{H}\mathrm{om}((- \otimes_A B) \circ V, V \circ (- \otimes_A B)))$$

By lemma 3.14 we see that $\mathcal{C}$ is closed under shifts. Also by a similar argument as in lemma 4.5 we see that $\mathcal{C}$ is closed under extensions and direct summands. Since $\mathcal{C}$ contains A the result follows. □

Recall that we use the notation $M^* = \mathcal{H}\mathrm{om}_A(M, A)$. Consider the module

$$V(M) \otimes_A B \otimes_A M^*$$

The composition

$$\begin{aligned} &V(M) \otimes_A B \otimes_A M^* \to \mathcal{H}\mathrm{om}_A(M, V(M) \otimes_A B) \\ &\cong \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M) \otimes_A B) \to \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B)) \end{aligned}$$

gives us a morphism

$$\psi : V(M) \otimes_A B \otimes_A M^* \to \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B))$$

Let $f \otimes b_1 \otimes g \in V(M) \otimes_A B \otimes_A M^*$. We see that

$$\psi(f \otimes b_1 \otimes g)(m_1 \otimes b_2) = \nu(f \otimes (b_1 \cdot g(m_1) \cdot b_2))$$

where $\nu : V(M) \otimes_A B \to V(M \otimes_A B)$ in the map defined in lemma 4.11. This implies that

$$\begin{aligned} &\psi(f \otimes b_1 \otimes g)(m_1 \otimes b_2)(m_2 \otimes b_3) \\ &= (-1)^{(|b_1|+|g|+|m_1|+|b_2|)\cdot|f|} \cdot \tau_B(b_1 \cdot g(m_1) \cdot b_2) \cdot f(m_2) \cdot b_3 \end{aligned} \tag{4.6}$$

Now consider the composition

$$\begin{aligned} V(M) \otimes_A B \otimes_A M^* \xrightarrow{1\otimes\tau_B\otimes 1} &V(M) \otimes_A \overline{B} \otimes_A M^* \\ &\cong \overline{M^*} \otimes_A \overline{\overline{B}} \otimes_A \overline{V(M)} \\ &= V(M) \otimes_A B \otimes_A M^* \end{aligned}$$

We denote this map by

$$\chi_1 : V(M) \otimes_A B \otimes_A M^* \to V(M) \otimes_A B \otimes_A M^*$$

Hence

$$\chi_1(f \otimes b \otimes g) = (-1)^{|f|\cdot|b|+|f|\cdot|g|+|b|\cdot|g|} \cdot g \otimes \tau_B(b) \otimes f$$

We also have a map

$$\chi_2 : \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B)) \to \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B))$$

given by

$$\chi_2(f) = V(f) \circ \phi_{M \otimes_A B}$$

Note that $\chi_1^2 = 1$ and $\chi_2^2 = 1$.

Lemma 4.12. (2.10 in [7]) *The morphism ψ commute with χ_i, i.e*

$$\psi \circ \chi_1 = \chi_2 \circ \psi$$

Proof. We first consider the composition $\psi \circ \chi_1$. Observe that

$$\psi \circ \chi_1(f \otimes b_1 \otimes g) = (-1)^{|f|\cdot|b|+|f|\cdot|g|+|b|\cdot|g|} \cdot \psi(g \otimes \tau_B(b) \otimes f)$$

Applying equation (4.6) we get

$$\begin{aligned}
&\psi \circ \chi_1(f \otimes b_1 \otimes g)(m_1 \otimes b_2)(m_2 \otimes b_3) \\
&= (-1)^{|f|\cdot|b_1|+|g|\cdot|m_1|+|g|\cdot|b_2|} \cdot \tau_B(\tau_B(b_1) \cdot f(m_1) \cdot b_2) \cdot g(m_2) \cdot b_3 \\
&= (-1)^t \cdot \tau_B(b_2) \cdot \tau_B(f(m_1)) \cdot b_1 \cdot g(m_2) \cdot b_3
\end{aligned}$$

where

$$t = |g| \cdot |m_1| + |g| \cdot |b_2| + |b_1| \cdot |b_2| + |b_1| \cdot |m_1| + |f| \cdot |b_2| + |m_1| \cdot |b_2|$$

Now consider $\chi_2 \circ \psi$. We have that

$$\chi_2 \circ \psi(f \otimes b_1 \otimes g) = V(\psi(f \otimes b_1 \otimes g)) \circ \phi_{M \otimes_A B}$$

Hence

$$\begin{aligned}
&\chi_2 \circ \psi(f \otimes b_1 \otimes g)(m_1 \otimes b_2)(m_2 \otimes b_3) \\
&= V(\psi(f \otimes b_1 \otimes g))(\phi_{M \otimes_A B}(m_1 \otimes b_2))(m_2 \otimes b_3) \\
&= (-1)^{s_1} \phi_{M \otimes_A B}(m_1 \otimes b_2)(\psi(f \otimes b_1 \otimes g)(m_2 \otimes b_3))
\end{aligned}$$

where $s_1 = |m_1| \cdot |f| + |m_1| \cdot |b_1| + |m_1| \cdot |g| + |b_2| \cdot |f| + |b_2| \cdot |b_1| + |b_2| \cdot |g|$. Furthermore

$$\begin{aligned}
&(-1)^{s_1} \cdot \phi_{M \otimes_A B}(m_1 \otimes b_2)(\psi(f \otimes b_1 \otimes g)(m_2 \otimes b_3)) \\
&= (-1)^{s_2} \cdot \tau_B(\psi(f \otimes b_1 \otimes g)(m_2 \otimes b_3)(m_1 \otimes b_2))
\end{aligned}$$

where $s_2 = |m_1| \cdot |m_2| + |m_1| \cdot |b_3| + |b_2| \cdot |m_2| + |b_2| \cdot |b_3|$. Also

$$\begin{aligned}
&(-1)^{s_2} \cdot \tau_B(\psi(f \otimes b_1 \otimes g)(m_2 \otimes b_3)(m_1 \otimes b_2)) \\
&= (-1)^{s_3} \cdot \tau_B(\tau_B(b_1 \cdot g(m_2) \cdot b_3) \cdot f(m_1) \cdot b_2)
\end{aligned}$$

where $s_3 = s_2 + |f| \cdot |b_1| + |f| \cdot |g| + |f| \cdot |m_2| + |f| \cdot |b_3|$. Finally

$$\begin{aligned}
&= (-1)^{s_3} \cdot \tau_B(\tau_B(b_1 \cdot g(m_2) \cdot b_3) \cdot f(m_1) \cdot b_2) \\
&= (-1)^t \cdot \tau_B(b_2) \cdot \tau_B(f(m_1)) \cdot b_1 \cdot g(m_2) \cdot b_3
\end{aligned}$$

and the result follows. □

In proposition 4.3 we showed that $V_k = [k] \circ V$ is a preduality functor. We want to find a version of lemma 4.12 with V replaced by V_k. First observe that we have a morphism

$$\begin{aligned}
&V_k(M) \otimes_A B \otimes_A M^* \cong (V(M) \otimes_A B \otimes_A M^*)[k] \\
&\xrightarrow{\psi[k]} \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B))[k] \cong \mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B))
\end{aligned}$$

which we denote by ψ^k. We also have a map

$$\chi_2^k : \mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B)) \to \mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B))$$

given by

$$\chi_2^k(\Sigma^k_{V(M\otimes_A B)} \circ f) = V_k(\Sigma^k_{V(M\otimes_A B)} \circ f) \circ \phi^k_{M\otimes_A B}$$

We want to find a map χ_1^k on $V_k(M) \otimes_A B \otimes_A M^*$ such that lemma 4.12 holds for ψ^k, χ_2^k and χ_1^k. A natural candidate is the composition

$$\begin{aligned} &V_k(M) \otimes_A B \otimes_A M^* \cong (V(M) \otimes_A B \otimes_A M^*)[k] \\ &\xrightarrow{\chi_1[k]} (V(M) \otimes_A B \otimes_A M^*)[k] \cong V_k(M) \otimes_A B \otimes_A M^* \end{aligned}$$

We denote this by $\chi_1'^k$. Explicitely it is given by

$$\chi_1'^k(\Sigma^k(f) \otimes b \otimes g) = (-1)^{|f|\cdot|b|+|f|\cdot|g|+|b|\cdot|g|} \cdot \Sigma^k(g) \otimes \tau_B(b) \otimes f$$

By lemma 4.12 we have that

$$\psi^k \circ \chi_1'^k = \chi_2'^k \circ \psi^k$$

where $\chi_2'^k$ is the composition

$$\begin{aligned} &\mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B)) \cong \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B))[k] \\ &\xrightarrow{\chi_2[k]} \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B))[k] \cong \mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B)) \end{aligned}$$

Explicitely $\chi_2'^k$ is given by

$$\chi_2'^k(\Sigma^k_{V(M\otimes_A B)} \circ f) = \Sigma^k_{V(M\otimes_A B)} \circ V(f) \circ \phi_{M\otimes_A B}$$

We compare this to $\chi_2^k(\Sigma^k_{V(M\otimes_A B)} \circ f) = V_k(\Sigma^k_{V(M\otimes_A B)} \circ f) \circ \phi^k_{M\otimes_A B}$. Recall that $\phi^k_{M\otimes_A B}$ is the composition

$$M \otimes_A B \xrightarrow{\phi_{M\otimes_A B}} VV(M \otimes_A B) \xrightarrow{(\alpha^k_{V(M\otimes_A B)[k]})^{-1}} V_k \circ V_k(M \otimes_A B)$$

(see 4.2), where

$$(\alpha^k_{V(M\otimes_A B)[k]})^{-1} = (-1)^{k(k-1)/2} \cdot \Sigma^k_{V(V(M\otimes_A B)[k])} \circ V(\Sigma^{-k}_{V(M\otimes_A B)[k]})$$

We also have that

$$\begin{aligned} V_k(\Sigma^k_{V(M\otimes_A B)} \circ f) &= (-1)^{k\cdot(|f|+k)} \cdot \Sigma^k_{V(M\otimes_A B)} \circ V(\Sigma^k_{V(M\otimes_A B)} \circ f) \circ \Sigma^{-k}_{V_k V_k(M\otimes_A B)} \\ &= (-1)^k \cdot \Sigma^k_{V(M\otimes_A B)} \circ V(f) \circ V(\Sigma^k_{V(M\otimes_A B)}) \circ \Sigma^{-k}_{V_k V_k(M\otimes_A B)} \end{aligned}$$

This implies that

$$\begin{aligned}
&\chi_2^k(\Sigma^k_{V(M\otimes_A B)} \circ f) \\
&= V_k(\Sigma^k_{V(M\otimes_A B)} \circ f) \circ \phi^k_{M\otimes_A B} \\
&= (-1)^{k(k+1)/2} \cdot \Sigma^k_{V(M\otimes_A B)} \circ V(f) \circ V(\Sigma^k_{V(M\otimes_A B)}) \circ V(\Sigma^{-k}_{V(M\otimes_A B)[k]}) \circ \phi_{M\otimes_A B} \\
&= (-1)^{k(k-1)/2} \cdot \Sigma^k_{V(M\otimes_A B)} \circ V(f) \circ \phi_{M\otimes_A B} \\
&= (-1)^{k(k-1)/2} \cdot \chi_2'^k(\Sigma^k_{V(M\otimes_A B)} \circ f)
\end{aligned}$$

We therefore define $\chi_1^k = (-1)^{k(k-1)/2} \cdot \chi_1'^k$, so

$$\chi_1^k(\Sigma^k(f) \otimes b \otimes g) = (-1)^{|f|\cdot|b|+|f|\cdot|g|+|b|\cdot|g|+k(k-1)/2} \cdot \Sigma^k(g) \otimes \tau_B(b) \otimes f$$

This gives us the following result

Proposition 4.13. *Let ψ^k be the composition*

$$\begin{aligned}
&V_k(M) \otimes_A B \otimes_A M^* \cong (V(M) \otimes_A B \otimes_A M^*)[k] \\
&\xrightarrow{\psi[k]} \mathcal{H}\mathrm{om}_B(M \otimes_A B, V(M \otimes_A B))[k] \cong \mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B))
\end{aligned}$$

where ψ is given in (4.6). Also let

$$\chi_1^k : V_k(M) \otimes_A B \otimes_A M^* \to V_k(M) \otimes_A B \otimes_A M^*$$

and

$$\chi_2^k : \mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B)) \to \mathcal{H}\mathrm{om}_B(M \otimes_A B, V_k(M \otimes_A B))$$

be given by

$$\chi_1^k(\Sigma^k(f) \otimes b \otimes g) = (-1)^{|f|\cdot|b|+|f|\cdot|g|+|b|\cdot|g|+k(k-1)/2} \cdot \Sigma^k(g) \otimes \tau_B(b) \otimes f$$

and

$$\chi_2^k(\Sigma^k_{V(M\otimes_A B)} \circ f) = V_k(\Sigma^k_{V(M\otimes_A B)} \circ f) \circ \phi^k_{M\otimes_A B}$$

Then ψ^k commutes with χ_i^k, i.e

$$\psi^k \circ \chi_1^k = \chi_2^k \circ \psi^k$$

5. THE N-CALABI-YAU COMPLETION

Consider the dg algebra

$$A^e = A \otimes_k A^{\mathrm{op}}$$

where A is a dg algebra. The multiplicative structure on A^e is given by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|a_2| \cdot b_1|} \cdot (a_1 \cdot a_2) \otimes (b_1 * b_2)$$

where $*$ is multiplication in A^{op}. The chain map

$$\tau : A \otimes_k A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes_k A$$

given by

$$\tau(a \otimes b) = (-1)^{|a| \cdot |b|} \cdot b \otimes a$$

will be an involution on A^e. Note that A is a right dg A^e-module via the action

$$a \cdot (a_1 \otimes a_2) = (-1)^{|a_2| \cdot |a_1| + |a_2| \cdot |a|} \cdot a_2 \cdot a \cdot a_1$$

We say that A is *homologically smooth* if it is a perfect dg A^e module. If A is an ordinary algebra then A is homologically smooth if and only if A has finite projective dimension as an A^e-module.

In this chapter we will define the n-Calabi-Yau-completion B of a homologically smooth finite dimensional algebra. We will show that B is a homologically smooth dg algebra and that B is quasi-isomorphic to $DV_n(B)$, where $V_n = [n] \circ V$ and V is the preduality functor on B^e induces from the involution. This will imply that the full subcategory $D_{fd}(B^e)$ of $D(B^e)$ is n-Calabi-Yau. In the last subsection we show that the preprojective algebra $\mathcal{P}_k(Q)$ is quasi-isomorphic to the 2-Calabi-Yau completion of the path algebra kQ.

5.1 Statement of the main result

Let A be a homologically smooth dg algebra over the field k. Let

$$V : \mathcal{C}_{dg}(A^e) \to \mathcal{C}_{dg}(A^e)^{\mathrm{op}}$$

be the preduality functor induces from the involution on A^e. We define the *inverse dualizing complex* of A to be

$$\Theta_A = p \circ DV(A)$$

where DV is the derived functor of V and p is the homotopically projective resolution functor in Theorem 3.25. Note that Θ_A is well defined up to homotopy equivalence. Now let n be an integer and

$$\theta_A = \Theta_A[n-1]$$

The *n-Calabi-Yau completion* of A is the tensor dg algebra

$$B = T_A(\theta_A) = A \oplus \theta_A \oplus (\theta_A \otimes_A \theta_A) \oplus ...$$

with differential acting componentwise on B. Note that B is well defined up to homotopy equivalence. We can consider A as a dg algebra concentrated in degree 0. With this identification we get a natural morphism of dg algebras

$$F : A \to B$$

Recall that

$$\phi'^n : 1 \to DV_n \circ DV_n$$

denote the induced natural transformation on $D(B^e)$ defined in subsection 4.3.

Theorem 5.1. (Theorem 4.8 in [7]) *Let A be a homologically smooth finite dimensional algebra, and let B be the n-Calabi-Yau completion of A. Then B is homologically smooth and we have an isomorphism in $D(B^e)$*

$$f : B \to DV_n(B)$$

satisfying

$$f = DV_n(f) \circ \phi'^n_B$$

Recall that $D_{fd}(B)$ is the subcategory of $D(B)$ consisting of modules with finite dimension in homology. One of the main reasons why Theorem 5.1 is important is due to the following result.

Theorem 5.2. (Lemma 3.4 in [7]) *Assume B is a homologically smooth dg algebra over the field k. For any dg module L and any dg module M in $D_{fd}(B)$, there is a canonical isomorphism*

$$\mathrm{Hom}_{D(B)}(L \otimes_B \Theta_B, M) \xrightarrow{\cong} \mathrm{Hom}_k(\mathrm{Hom}_{D(B)}(M, L), k)$$

In particular if Θ_B is isomorphic to $B[-n]$ in $D(B^e)$, then $D_{fd}(B)$ is n-Calabi-Yau as a triangulated category.

Note that $\Theta_B = DV_n(B)[-n]$. Hence if B is the n-Calabi-Yau completion of some finite dimensional homologically smooth algebra A, then $D_{fd}(B)$ is n-Calabi-Yau.

5.2 Proof of the main result

In this section we want to give a proof of Theorem 5.1. But first we need some preparation.

Lemma 5.3. *Let C be a dg A-bimodule. Let*

$$T_A(C) = A \oplus C \oplus (C \otimes_A C) \oplus$$

be the tensor algebra. Then $T_A(C)$ is a dg algebra and we have an exact sequence of right dg $T_A(C)^e$ modules

$$0 \to T_A(C) \otimes_A C \otimes_A T_A(C) \xrightarrow{\alpha} T_A(C) \otimes_A T_A(C) \xrightarrow{m} T_A(C) \to 0$$

where $\alpha(b \otimes x \otimes c) = b \otimes (x \cdot c) - (b \cdot x) \otimes c$ and $m(b \otimes c) = b \cdot c$ are morphism of dg $T_A(C)^e$-modules.

Proof. It is obvious that $T_A(C)$ is a dg algebra and that α and m are morphism of dg A^e-modules. For exactness observe that we have a map

$$s : T_A(C) \to T_A(C) \otimes_A T_A(C)$$

given by

$$s(b) = b \otimes 1$$

Note that s is a right inverse for m, i.e

$$m \circ s = 1$$

We also have a map

$$t : T_A(C) \otimes_A T_A(C) \to T_A(C) \otimes_A C \otimes_A T_A(C)$$

given by

$$\begin{aligned}
&t((b_1 \otimes ... \otimes b_k) \otimes (c_1 \otimes ... \otimes c_l)) \\
&= (b_1 \otimes ... \otimes b_k) \otimes c_1 \otimes (c_2 \otimes ... \otimes c_l) + (b_1 \otimes ... \otimes b_k \otimes c_1) \otimes c_2 \otimes (c_3 \otimes ... \otimes c_l) \\
&+ ... + ((b_1 \otimes ... \otimes b_k \otimes c_1 \otimes ... \otimes c_{l-1}) \otimes c_l \otimes 1
\end{aligned}$$

It is easy to see that

$$t \circ \alpha = 1$$

and

$$\alpha \circ t + s \circ m = 1$$

Hence the result follows. $\square$

We will now assume A is a finite dimensional homologically smooth algebra.

Lemma 5.4. *We have a projective resolution of A*

$$0 \to P_k \to ... \to P_0 \to A \to 0$$

of finite dimensional A^e-bimodules such that $P_0 = A^e$.

Proof. We have an A^e-bimodule epimorphism

$$A^e \xrightarrow{m} A$$

given by $m(a_1 \otimes a_2) = a_1 \cdot a_2$. Observe that $\mathrm{Ker}(m)$ has finite projective dimension since A has finite projective dimension. Picking a projective resolution of $\mathrm{Ker}(m)$ and append A^e to the end gives us the required projective resolution for A. □

Now pick a projective resolution $P = P_k \to ... \to P_0$ of A as in lemma 5.4. Consider the dg A^e-module

$$DV(A) = \overline{\mathcal{H}\mathrm{om}_{A^e}(P, A^e)} = \overline{\mathrm{Hom}_{A^e}(P_0, A^e)} \to ... \to \overline{\mathrm{Hom}_{A^e}(P_k, A^e)}$$

Since $\overline{\mathrm{Hom}_{A^e}(P_i, A^e)}$ is projective for $i = 0, ..., k$ we get that $\overline{\mathcal{H}\mathrm{om}_{A^e}(P, A^e)}$ is homotopically projective. Hence

$$\Theta_A = p \circ DV(A) = \overline{\mathcal{H}\mathrm{om}_{A^e}(P, A^e)}$$

This implies that

$$\theta_A = \Theta_A[n-1] = \overline{\mathrm{Hom}_{A^e}(P_0, A^e)} \to ... \to \overline{\mathrm{Hom}_{A^e}(P_k, A^e)}$$

concentrated in degrees $-n+1$, $-n+2$,..,$-n+1+k$. Note that θ_A is strictly perfect since it is a bounded complex of finite dimensional projectives. Let $B = T_A(\theta_A)$ be the n-Calabi-Yau completion of A. Lemma 5.3 tells us that we have an exact sequence

$$0 \to B \otimes_A \theta_A \otimes_A B \xrightarrow{\alpha} B \otimes_A B \xrightarrow{m} B \to 0 \tag{5.1}$$

We want to show that B is homologically smooth, but first we need some preparation. Note that the dg algebra morphism $F : A \to B$ gives us a dg algebra morphism

$$F^e : A^e \to B^e$$

given by

$$F^e(a_1 \otimes a_2) = F(a_1) \otimes F(a_2)$$

In this way we can consider B^e as a dg A^e-module.

Lemma 5.5. *Let M be a right dg A^e module. We have a natural isomorphism of dg modules*

$$M \otimes_{A^e} B^e \xrightarrow{\cong} B \otimes_A M \otimes_A B$$

sending $m \otimes (b_1 \otimes b_2)$ to $(-1)^{|b_2|\cdot|b_1|+|b_2\cdot|m|} \cdot b_2 \otimes m \otimes b_1$.

Proof. This is a straightforward calculation □

Lemma 5.5 tells us that

$$B \otimes_A \theta_A \otimes_A B \cong \theta_A \otimes_{A^e} B^e$$

and

$$B \otimes_A B \cong A \otimes_{A^e} B^e$$

With this identification we see that the morphism α is given by

$$\alpha(x \otimes (b_1 \otimes b_2)) = (x \cdot b_1) \otimes b_2 - (-1)^{(|b_1|+|b_2|)\cdot|x|} \cdot b_1 \otimes (b_2 \cdot x)$$

Lemma 5.6. *Let M and N be respectively projective left and right A-modules. Then $M \otimes_k N$ is a projective A^e-module.*

Proof. There exists positive integers m_1 and m_2 such that M is a direct summand of A^{m_1} and N is a direct summand of A^{m_2}. Hence $M \otimes_k N$ is a direct summand of

$$A^{m_1} \otimes_k A^{m_2} \cong (A^e)^{m_1 \cdot m_2}$$

Therefore $M \otimes_k N$ is a projective A^e-module. □

Lemma 5.7. *Let M and N be (ordinary) A^e-modules. Assume that M and N are projective as left and as right A-modules. Then*

$$M \otimes_A N$$

is projective as a left and as a right A-module.

Proof. Let X and Y be right A-modules, and let $f : M \otimes_A N \to Y$ and $g : X \to Y$ be morphisms of right A-modules. Furthermore assume that g is surjective. We want to show that f factors through g. Let

$$\eta_X : \mathrm{Hom}_A(M \otimes_A N, X) \xrightarrow{\cong} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, X))$$

and

$$\eta_Y : \mathrm{Hom}_A(M \otimes_A N, Y) \xrightarrow{\cong} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, Y))$$

denote the hom-tensor adjunctions, where the elements in the hom sets are morphisms of right modules. Consider the diagram

$$\begin{tikzcd} & \mathrm{Hom}_A(N,X) \arrow[d, "g\circ -"] \\ M \arrow[ur, dotted, "h"] \arrow[r, "\eta_Y(f)"'] & \mathrm{Hom}_A(N,Y) \end{tikzcd}$$

Since N is projective the map

$$g \circ - : \mathrm{Hom}_A(N,X) \to \mathrm{Hom}_A(N,Y)$$

is surjective. Therefore since M is projective there exists a morphism h making the diagram commute. This implies that the diagram

$$\begin{tikzcd} & X \arrow[d, "g"] \\ M \otimes_A N \arrow[ur, "\eta_X^{-1}(h)"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes, and hence $M \otimes_A N$ is a projective right A-module. A similar argument shows that $M \otimes_A N$ is a projective left A-module. □

Lemma 5.8. *Let M be a projective A^e-module. Then M is a projective right A-module and a projective left A-module.*

Proof. There exists a positive integer m such that M is a direct summand of $(A^e)^m$. Furthermore there exists bijections

$$r : A^e \xrightarrow{\cong} A^d$$

and

$$l : A^e \xrightarrow{\cong} A^d$$

given by

$$r((k_1 \cdot e_1 + ... + k_d \cdot e_d) \otimes b) = (k_1 \cdot b, k_2 \cdot b, ..., k_d \cdot b)$$

and

$$l(a \otimes (k_1 \cdot e_1 + ... + k_d \cdot e_d)) = (k_1 \cdot a, k_2 \cdot a, ..., k_d \cdot a)$$

where d is the dimension of A, $k_i \in k$ for $i = 1, ..., d$, and $a, b \in A$. Using these bijections we see that M is a direct summand of $A^{d \cdot m}$ as a right module and as a left module. The result follows. □

Lemma 5.9. *B^e is a homotopically projective A^e-module.*

Proof. From lemma 5.8 we know that θ_A is a bounded complex of projective right and projective left A-modules. Hence

$$\theta_A \otimes_A \ldots \otimes_A \theta_A$$

is a bounded complex of projective right and projective left A-modules by lemma 5.7. Lemma 5.6 implies that

$$(\theta_A \otimes_A \ldots \otimes_A \theta_A) \otimes_k (\theta_A \otimes_A \ldots \otimes_A \theta_A)$$

is a bounded complex of projective A^e-modules, and is therefore homotopically projective. Since B^e is a sum of such terms we get that it is homotopically projective. □

Lemma 5.10. *$\theta_A \otimes_{A^e} B^e$ is a strictly perfect dg module and $A \otimes_{A^e} B^e$ is a perfect dg module.*

Proof. Since θ_A is a strictly perfect dg A^e-module and $- \otimes_{A^e} B^e$ is a dg functor taking A^e to B^e we get that $\theta_A \otimes_{A^e} B^e$ is a strictly perfect B^e-module (see 3.9).

For the the second part note that A is a perfect dg A^e-modules. Hence it is enough to show that the functor $- \otimes_{A^e} B^e$ takes perfect modules to perfect modules. First observe that B^e is homotopically projective, so $- \otimes_{A^e} B^e$ induces a well defined triangle functor

$$- \otimes_{A^e} B^e : D(A^e) \to D(B^e)$$

Consider the full subcategory $\mathcal{C}$ of $D(A^e)$ consisting of objects M such that $M \otimes_{A^e} B^e$ is perfect. This is a triangulated category since $- \otimes_{A^e} B^e$ is a triangle functor. It is also obviously closed under direct summands and contains the object A^e since $A^e \otimes_{A^e} B^e = B^e$ is perfect. Hence $\mathcal{C}$ contains $\mathrm{perf}(A)$, and we are done. □

Corollary 5.11. *B is a homologically smooth dg algebra*

Proof. We need to show that B is a perfect B^e-module. The exact sequence given in 5.1 implies that we have a quasi-isomorphism

$$\mathrm{Cone}(\alpha) \simeq B$$

Since α is a morphism between perfect modules we have that $\mathrm{Cone}(\alpha)$ is perfect. Therefore B is a perfect, and we are done. □

Now consider the dg module $V_{n-1}(\theta_A)$. This is given by

$$V_{n-1}(\theta_A) = \overline{\mathrm{Hom}_{A^e}(\overline{\mathrm{Hom}_{A^e}(P_k, A^e)}, A^e)} \to \ldots \to \overline{\mathrm{Hom}_{A^e}(\overline{\mathrm{Hom}_{A^e}(P_0, A^e)}, A^e)}$$

concentrated in degrees $-k$,...,0. Since P_i is projective we have an isomorphism of A^e-modules

$$P_i \cong \overline{\mathrm{Hom}_{A^e}(\overline{\mathrm{Hom}_{A^e}(P_i, A^e)}, A^e)}$$

This induces an isomorphism

$$V_{n-1}(\theta_A) \cong P$$

where

$$P = P_k \to \ldots \to P_0$$

Since P is a projective resolution of A we get a quasi-isomorphism

$$p : V_{n-1}(\theta_A) \to A \tag{5.2}$$

This induces a quasi-isomorphism

$$V_{n-1}(\theta_A) \otimes_{A^e} B^e \xrightarrow{p\otimes 1} A \otimes_{A^e} B^e$$

where we use the fact that B^e is homotopically projective. Furthermore the map

$$V(\theta_A) \otimes_{A^e} B^e \to V(\theta_A \otimes_{A^e} B^e)$$

in lemma 4.11 is an isomorphism since θ_A is strictly perfect. We therefore have an isomorphism

$$\nu^{n-1} : V_{n-1}(\theta_A) \otimes_{A^e} B^e \xrightarrow{\cong} V_{n-1}(\theta_A \otimes_{A^e} B^e)$$

Hence we have a diagram

$$\begin{tikzcd}
\theta_A \otimes_{A^e} B^e \arrow[r, "\alpha"] \arrow[dd, "="] & A \otimes_{A^e} B^e \\
 & V_{n-1}(\theta_A) \otimes_{A^e} B^e \arrow[u, "p\otimes 1"'] \arrow[d, "\nu^{n-1}"] \\
\theta_A \otimes_{A^e} B^e \arrow[r, dotted, "\alpha'"] & V_{n-1}(\theta_A \otimes_{A^e} B^e)
\end{tikzcd} \tag{5.3}$$

Note that $\phi_{\theta_A \otimes_{A^e} B^e}$ is an isomorphism since $\theta_A \otimes_{A^e} B^e$ is strictly perfect. The idea for the remaining part of the proof is the following. We want to find a morphism α' which is antisymmetric and makes (5.3) commute in $D(B^e)$. From proposition 4.4 we will then get a V_n symmetric quasi-isomorphism $\mathrm{Cone}(\alpha') \to V_n(\mathrm{Cone}(\alpha'))$. Using the commutativity of (5.3) and the fact that $\mathrm{Cone}(\alpha')$ is quasi-isomorphic to B we then finally construct the morphism f in Theorem 5.1.

In order to construct the morphism α' we consider the commutative diagram

$$\begin{tikzcd}
 & \mathcal{H}\mathrm{om}_{B^e}(\theta_A \otimes_{A^e} B^e, V_{n-1}(\theta_A \otimes_{A^e} B^e)) \\
V_{n-1}(\theta_A) \otimes_{A^e} B^e \otimes_{A^e} \theta_A^* \arrow[r] \arrow[ur, "\psi^{n-1}"] \arrow[dr, "\Psi^{n-1}"'] & \mathcal{H}\mathrm{om}_{B^e}(\theta_A \otimes_{A^e} B^e, V_{n-1}(\theta_A) \otimes_{A^e} B^e) \arrow[u, "\nu^{n-1} \circ -"'] \arrow[d, "(p \otimes 1) \circ -"] \\
 & \mathcal{H}\mathrm{om}_{B^e}(\theta_A \otimes_{A^e} B^e, A \otimes_{A^e} B^e)
\end{tikzcd} \tag{5.4}$$

The horisontal morphism is the composition of the maps given in lemma 4.7 and lemma 4.9. The vertical morphisms are induced from the morphisms

$$\nu^{n-1} : V_{n-1}(\theta_A) \otimes_{A^e} B^e \to V_{n-1}(\theta_A \otimes_{A^e} B^e)$$

and

$$p \otimes 1 : V_{n-1}(\theta_A) \otimes_{A^e} B^e \to A \otimes_{A^e} B^e$$

The diagonal morphisms are the composition of the horisontal and the vertical morphisms.

Recall from proposition 4.13 that we can talk about antisymmetric elements in $V_{n-1}(\theta_A) \otimes_{A^e} B^e \otimes_{A^e} \theta_A^*$ (i.e elements x satisfying $\chi_1^{n-1}(x) = -x$), and that the morphism

$$V_{n-1}(\theta_A) \otimes_{A^e} B^e \otimes_{A^e} \theta_A^* \xrightarrow{\psi^{n-1}} \mathcal{H}\mathrm{om}_{B^e}(\theta_A \otimes_{A^e} B^e, V_{n-1}(\theta_A \otimes_{A^e} B^e))$$

takes antisymmetric elements to antisymmetric morphisms. Hence it is enough to find a antisymmetric element that maps to α under the morphism

$$V_{n-1}(\theta_A) \otimes_{A^e} B^e \otimes_{A^e} \theta_A^* \xrightarrow{\Psi^{n-1}} \mathcal{H}\mathrm{om}_{B^e}(\theta_A \otimes_{A^e} B^e, A \otimes_{A^e} B^e)$$

in diagram 5.4.

At the end of 4.8 in [7] Keller claims that the element $id \otimes c$ maps to the morphism

$$x \to x \otimes 1$$

and its transpose conjugate $\chi_1^{n-1}(id \otimes c)$ maps to the morphism

$$x \to 1 \otimes x$$

under Ψ^{n-1}, where $c \in \theta_A \otimes_{A^e} \theta_A^*$ is the Casimir element (i.e the preimage of $1 \in \mathcal{H}\mathrm{om}_{A^e}(\theta_A, \theta_A)$ under the morphism in lemma 4.7), and where we use the identification

$$\mathcal{H}\mathrm{om}_{B^e}(\theta_A \otimes_{A^e} B^e, A \otimes_{A^e} B^e) \cong \mathcal{H}\mathrm{om}_{B^e}(\theta_A, A \otimes_{A^e} B^e)$$

It is then easy to see that the antisymmetric element $id \otimes c - \chi_1^{n-1}(id \otimes c)$ maps to α'. Unfortunately, due to time limits we haven't been able to verify that this is true.

Now assuming the existence of such a morphism α', we get a V_n-symmetric quasi-isomorphism

$$f : \mathrm{Cone}(\alpha') \to V_n(\mathrm{Cone}(\alpha'))$$

from proposition 4.4. Note that $V_{n-1}(\theta_A \otimes_{A^e} B^e)$ is strictly perfect since $\theta_A \otimes_{A^e} B^e$ is strictly perfect. Hence $\mathrm{Cone}(\alpha')$ is strictly perfect, and therefore $V_n(\mathrm{Cone}(\alpha'))$ is also strictly perfect. Since strictly perfect modules are homotopically projective (lemma 3.22), f must be a quasi-isomorphism

$$f : \mathrm{Cone}(\alpha') \to DV_n(\mathrm{Cone}(\alpha'))$$

satisfying

$$f = DV_n(f) \circ \phi'^n_{\mathrm{Cone}(\alpha')}$$

by lemma 4.6. Theorem 5.1 follows from the quasi-isomorphism

$$B \simeq \mathrm{Cone}(\alpha')$$

5.3 Relation to the preprojective algebra of a quiver

Let Q be a finite quiver without cycles. Let kQ have a right $kQ^e = kQ \otimes_k kQ^{\mathrm{op}}$ module structure given by

$$x \cdot (y \otimes z) = z \cdot x \cdot y$$

We want to consider the 2-Calabi-Yau completion $\Pi_2(kQ)$ of kQ. In order for the results in the previous section to hold we need to show that kQ is homologically smooth. For this we will use the exact sequence in lemma 2.10 with $M = kQ$

Lemma 5.12. *We have a projective resolution of right $kQ^e = kQ \otimes_k kQ^{\mathrm{op}}$ modules*

$$0 \to \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha} kQ \xrightarrow{\partial - \epsilon} \bigoplus_{i \in Q_0} kQe_i \otimes_k e_i kQ \xrightarrow{\zeta} kQ \to 0$$

where ∂,ϵ and ζ is the same as in lemma 2.10. In particular, kQ is homologically smooth.

Proof. We have already shown that the sequence is exact. Also $kQe_i \otimes e_j kQ$ has a canonical right kQ^e module structure given by

$$(m \otimes n)(x \otimes y) = y \cdot m \otimes n \cdot x$$

for $x \otimes y \in kQ^e$ and $m \otimes n \in kQe_i \otimes e_j kQ$. With this structure it is easy to see that ζ and $\partial - \epsilon$ are morphisms of kQ^e modules. It therefore only remain to show that

$$\bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha} k$$

and

$$\bigoplus_{i \in Q_0} kQe_i \otimes_k e_i kQ$$

are projective kQ^e modules, and for this it is sufficient to show that $kQe_i \otimes e_j kQ$ is a projective kQ^e module for $i, j \in Q_0$. Now we have an epimorphism

$$p : kQ^e \to kQe_i \otimes_k e_j kQ$$

given by $p(x \otimes y) = y \cdot e_i \otimes e_j \cdot x$. We also have a natural inclusion map

$$i : kQe_i \otimes_k e_j kQ \to kQ \otimes_k kQ^{\mathrm{op}} = kQ^e$$

given by

$$i(x \otimes y) = (y \otimes x)$$

and the composition $p \circ i$ is the identity map on $kQe_i \otimes_k e_j kQ$. Hence $kQe_i \otimes e_j kQ$ is a direct summand of kQ^e, and is therefore projective. $\square$

Recall that we have an isomorphism

$$\mathcal{P}_k(Q) \cong T_{kQ}(\theta)$$

where $\theta = \mathrm{Ext}^1_{kQ}(D(kQ), kQ)$ and $\mathcal{P}_k(Q)$ is the preprojective algebra. We now assume that the underlying graph of Q is not Dynkin. Our main goal is to show that we have a quasi-isomorphism of dg algebras

$$\mathcal{P}_k(Q) \to \Pi_2(kQ)$$

where we consider $\mathcal{P}_k(Q)$ as a dg algebra concentrated in degree 0.

Consider the projective resolution of $D(kQ)$ in lemma 2.10 given by

$$0 \to \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha} D(kQ) \xrightarrow{\partial - \epsilon} \bigoplus_{i \in Q_0} kQe_i \otimes_k e_i D(kQ) \xrightarrow{\zeta} D(kQ) \to 0$$

Since $D(kQ)$ is a kQ-bimodule we get that

$$\bigoplus_{\alpha\in Q_1} kQe_{t\alpha}\otimes_k e_{s\alpha}D(kQ)$$

and

$$\bigoplus_{i\in Q_0} kQe_i\otimes_k e_iD(kQ)$$

are kQ-bimodules, and therefore also left kQ^e-modules. It is easy to see that ζ and $\partial-\epsilon$ will be morphisms of left kQ^e modules. Applying $\mathrm{Hom}_{kQ}(-,kQ)$ we get the complex

$$\mathrm{Hom}_{kQ}(\bigoplus_{i\in Q_0} kQe_i\otimes_k e_iD(kQ),kQ)\xrightarrow{-\circ(\partial-\epsilon)}\mathrm{Hom}_{kQ}(\bigoplus_{\alpha\in Q_1} kQe_{t\alpha}\otimes_k e_{s\alpha}D(kQ),kQ)$$

of right kQ^e-modules. We denote this complex by C'. Note that

$$H^0(C')=\mathrm{Hom}_{kQ}(D(kQ),kQ)=0$$

since there are no morphisms from injectives to projectives when Q is non-Dynkin. Also note that

$$H^1(C')=\mathrm{Ext}^1_{kQ}(D(kQ_{kQ}),kQ)$$

Hence we get a quasi isomorphism

$$s:C\to\mathrm{Ext}^1_{kQ}(D(kQ_{kQ}),kQ)$$

where $C=C'[1]$. We want to show that this lifts to a quasi-isomorphism of dg algebras

$$s:T_{kQ}(C)\to T_{kQ}(\mathrm{Ext}^1_{kQ}(D(kQ_{kQ}),kQ))$$

Lemma 5.13. *Let X and Y be a finite dimensional left kQ modules. We then have a natural map*

$$\eta:\mathrm{Hom}_{kQ}(X,kQ)\otimes_{kQ}Y\to\mathrm{Hom}_{kQ}(X,Y)$$

given by $\eta(f\otimes y)(x)=f(x)\cdot y$. If X is projective then η is an isomorphism.

Proof. It is easy to see that η is well defined and natural. Also if $X=kQ$ then η is just the identity map on Y. Naturality of η makes it into an isomorphism when X is a sum of direct summands of kQ, and hence η is an isomorphism when X is projective. □

Lemma 5.14. *Let $\theta=\mathrm{Ext}^1_{kQ}(D(kQ_{kQ}),kQ)$ and let n a positive integer. Then the map*

$$s\otimes s\otimes s\otimes\ldots\otimes s:C\otimes_{kQ}C\otimes_{kQ}\ldots\otimes_{kQ}C\to\theta\otimes_{kQ}\theta\otimes_{kQ}\ldots.\otimes_{kQ}\theta$$

is a quasi isomorphism, where the tensor product is taken n times.

Proof. We show this by induction on n. The result is obviously true for $n = 1$, so we assume that it holds for $n-1$. Note that we have the identity

$$s \otimes s \otimes s \otimes ... \otimes s = (s \otimes 1 \otimes 1 \otimes ... \otimes 1) \circ (1 \otimes s \otimes s \otimes ... \otimes s)$$

Also, C is homotopically projective since it is a bounded complex with projective components. So tensoring with C preserves quasi isomorphisms, and in particular we have that

$$1 \otimes s \otimes s \otimes ... \otimes s : C \otimes_{kQ} C \otimes_{kQ} ... \otimes_{kQ} C \to C \otimes_{kQ} \theta \otimes_{kQ} .... \otimes_{kQ} \theta$$

is a quasi isomorphism by the induction step. Hence it only remains to show that

$$s \otimes 1_M : C \otimes_{kQ} M \to \theta \otimes_{kQ} M$$

is a quasi isomorphism, where $M = \theta \otimes_{kQ} .... \otimes_{kQ} \theta$. The chain complex $C \otimes_{kQ} M$ is given by

$$... \to 0 \to \mathrm{Hom}_{kQ}(P_0, kQ) \otimes_{kQ} M \xrightarrow{(-\circ f)\otimes 1} \mathrm{Hom}_{kQ}(P_1, kQ) \otimes_{kQ} M \to 0...$$

concentrated in degree -1 and 0, where

$$P_0 = \bigoplus_{i \in Q_0} kQe_i \otimes_k e_i D(kQ)$$

and

$$P_1 = \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha} D(kQ)$$

and $f = \partial - \epsilon$. Since tensor product is right exact we get that

$$\mathrm{coker}((- \circ f) \otimes 1) = \theta \otimes_{kQ} M$$

so $s \otimes 1_M$ is still an isomorphism in homology in degree 0. Hence it only remains to show that the map $(- \circ f) \otimes 1$ is mono. Note that we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{kQ}(P_0, kQ) \otimes_{kQ} M \arrow[r, "(- \circ f) \otimes 1"] \arrow[d, "\cong"] & \mathrm{Hom}_{kQ}(P_1, kQ) \otimes_{kQ} M \arrow[d, "\cong"] \\
\mathrm{Hom}_{kQ}(P_0, M) \arrow[r, "- \circ f"] & \mathrm{Hom}_{kQ}(P_1, M)
\end{tikzcd}$$

where the vertical maps are the isomorphisms in lemma 5.13. Note also that the kernel of the map

$$- \circ f : \mathrm{Hom}_{kQ}(P_0, M) \to \mathrm{Hom}_{kQ}(P_1, M)$$

is $\mathrm{Hom}_{kQ}(D(kQ), M)$. Now we have that $\mathrm{Hom}_{kQ}(D(kQ), M) = 0$ since $M = \theta \otimes_{kQ} .... \otimes_{kQ} \theta$ is preprojective, Q is non-Dynkin, and $D(kQ)$ is injective. Hence $- \circ f$ is mono, and the result follows. $\square$

Lemma 5.14 implies that we have a natural quasi isomorphism

$$s : T_{kQ}(C) \to \mathcal{P}_k(Q)$$

of dg algebras. Consider the resolution of kQ

$$0 \to \bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}kQ \xrightarrow{\partial - \epsilon} \bigoplus_{i \in Q_0} kQe_i \otimes_k e_i kQ \xrightarrow{\zeta} kQ \to 0$$

of projective right kQ^e modules given in lemma 5.12. Applying $\mathrm{Hom}_{kQ^e}(-, kQ^e)$ to it we get a complex of left kQ^e modules Θ' given by

$$\mathrm{Hom}_{kQ^e}(\bigoplus_{i \in Q_0} kQe_i \otimes_k e_i kQ, kQ^e) \xrightarrow{-\circ(\partial - \epsilon)} \mathrm{Hom}_{kQ^e}(\bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}kQ, kQ^e)$$

in degree 0 and 1, and with 0 in all other components. Now let $\Theta = \overline{\Theta'[1]}$ (see 4.1). Since Θ is homotopically projective we have that

$$\Pi_2(kQ) = T_{kQ}(\Theta)$$

Hence it only remains to show that Θ is isomorphic to C

Lemma 5.15. *There is an isomorphism $C \cong \Theta$ of right kQ^e chain complexes*

Proof. Let e_i and e_j be two idempotents of kQ correspond to vertex i and j. We have a natural isomorphism

$$\mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_j kQ, kQ \otimes_k kQ) \cong \mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_j kQ, \mathrm{Hom}_k(D(kQ), kQ))$$

by lemma 2.4. Now the hom-tensor adjunction gives us an isomorphism

$$\mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_j kQ, \mathrm{Hom}_k(D(kQ), kQ)) \cong \mathrm{Hom}_{kQ}(kQe_i \otimes_k e_j kQ \otimes_{kQ} D(kQ), kQ)$$

We also have that

$$kQe_i \otimes_k e_j kQ \otimes_{kQ} D(kQ) \cong kQe_i \otimes_k e_j D(kQ)$$

Putting all these isomorphisms together we get

$$\mathrm{Hom}_{kQ^e}(kQe_i \otimes_k e_j kQ, kQ \otimes_k kQ) \cong \mathrm{Hom}_{kQ}(kQe_i \otimes_k e_j D(kQ), kQ)$$

This induces isomorphisms

$$\mathrm{Hom}_{kQ^e}(\bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}kQ, kQ^e) \cong \mathrm{Hom}_{kQ}(\bigoplus_{\alpha \in Q_1} kQe_{t\alpha} \otimes_k e_{s\alpha}D(kQ), kQ)$$

and

$$\mathrm{Hom}_{kQ^e}(\bigoplus_{i \in Q_0} kQe_i \otimes_k e_i kQ, kQ^e) \cong \mathrm{Hom}_{kQ}(\bigoplus_{i \in Q_0} kQe_i \otimes_k e_i D(kQ), kQ)$$

This gives us a isomorphism of C and Θ as graded modules. It is also not hard to see that this map commutes with the differential, and hence C and Θ are isomorphic as chain complexes. $\square$

Putting all this together gives us the following result.

Theorem 5.16. *We have a quasi-isomorphism*

$$\Pi_2(kQ) \to \mathcal{P}_k(Q)$$

of dg algebras

There is a well-known result (see 3.10 in [9]) stating that a quasi-isomorphism

$$f : A \to B$$

of dg algebras A and B induces an equivalence

$$\mathbb{L}F : D(A) \to D(B)$$

where $\mathbb{L}F$ is the left derived functor of the induction functor

$$F = - \otimes_A B : \mathcal{H}A \to \mathcal{H}B$$

This means that $\mathbb{L}F = F \circ p$, where p is the homotopically projective resolution functor in Theorem 3.25. If we apply this to the case $A = \Pi_2(kQ)$, $B = \mathcal{P}_k(Q)$ and f is the quasi-isomorphism in Theorem 5.16, we get the following result

Theorem 5.17. *We have an equivalence of categories*

$$D(\Pi_2(kQ)) \cong D(\mathcal{P}_k(Q))$$

This restricts to an equivalence

$$D_{fd}(\Pi_2(kQ)) \cong D_{fd}(\mathcal{P}_k(Q))$$

In particular the category $D_{fd}(\mathcal{P}_k(Q))$ is 2-Calabi-Yau.

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