University Bonn — WS 2008 – 2009	Prof. Dr. Hamenstädt
Differential geometry	
Exercises 5	18th november 2008

5.1. Consider the exercise 4.2 a): prove that the projection $\pi_x : (x, y) \mapsto x$ onto the first coordinate of $\mathbb{R}^2$ descends to a map π' on the quotient $Q = \mathbb{R}^2/\mathbb{Z}$. Prove then that π' defines a vector bundle which is non-orientable.

5.2. A finite graph is a topological space which is obtained from the disjoint union of a finite number of closed intervals by identifications of some of the endpoints. The image of an interval in the quotient space is called an *edge* of the graph. For example, a triangle is a graph with 3 edges cyclically identified.

a) Define a vector bundle over a finite graph G.

b) Show that an oriented vector bundle over a finite graph is trivial.

c) A *cycle* in a graph is a subset of the graph which is homeomorphic to the circle. For example, a triangle is a cycle. Show that a vector bundle over a graph which does not contain any cycle is trivial.

5.3. Let $(e_0, \dots, e_n)$ be the canonical base for $\mathbb{R}^{n+1}$, let $S^n \subset \mathbb{R}^{n+1}$ be the sphere $\{(x_0, \dots, x_n) \mid \sum_i x_i^2 = 1\}$, and $E \subset \mathbb{R}^{n+1}$ be the subspace generated by the elements $e_1, \dots, e_n$. Consider the *stereographic projection* to be the map ϕ that associates to $x \in S^n \setminus \{e_0\}$ the point $\phi(x) \in E$ which is the intersection of the line passing through x and e_0 with E. Using this map:

a) Find an altas for TS^n consisting in two charts.

b) Construct a smooth section of TS^n that vanishes at only one point.

5.4. Let $E \to M$ and $F \to M$ be smooth vector bundles over a connected manifold. Show:

a) If E, F are orientable, then the same holds true for $E \oplus F$.

b) If E is not orientable, then there is a curve $c : [0, 1] \to M$ with the following properties.

i) $c(0) = c(1)$.

ii) If $c(s) = c(t)$ for $s < t$ then $s = 0, t = 1$.

iii) The restriction of E to $c[0, 1]$ is non-orientable.

c) Can we find $E \to M$ and $F \to M$ non-orientable such that $E \oplus F$ is non-orientable?