Basic Notions in Algebra – Exercise No. 9

1. Let $F \subset E$ be fields such that E/F is a cyclic extension, that is, a Galois extension with cyclic Galois group $G = Gal(E/F)$ of order n. Let $\sigma \in G$ be a generator (so that $G = \{1, \sigma, \ldots, \sigma^{n-1}\}$).

   Let $\gamma \in F$ be a non-zero element of F, and let D_γ be an n-dimensional vector space over E with basis $u_0, u_1, \ldots, u_{n-1}$. Define multiplication on D_γ by the following rules:

   $$u_i u_j = \begin{cases} u_{i+j} & \text{if } i+j < n \\ \gamma u_{i+j-n} & \text{if } i+j \geq n \end{cases} \qquad \text{and} \qquad u_i b = \sigma^i(b) u_i \text{ (where } b \in E)$$

   so that $b u_i b' u_j = b\sigma^i(b') u_i u_j$ and we extend by linearity.

   (a) Show that D_γ is an (associative) algebra over F, that E can be embedded in D_γ and that $[D_\gamma : F] = n^2$.

   (b) Show that D_γ is central over F (that is, $Z(D_\gamma) = F$ via the embedding of E in D_γ).

   (c) We say that γ is a *norm* of an element $c \in E$ if we have $\gamma = c\sigma(c)\sigma^2(c) \cdot \cdots \cdot \sigma^{n-1}(c)$.
   Show that if γ is a norm then $D_\gamma \simeq D_1$ are isomorphic as algebras over F.

   (d) Show that D_1 has zero divisors.

2. Let E, F, σ be as in the previous question. For each $b \in E$, let $\bar{b}$ be the diagonal matrix in $M_n(E)$ defined as

   $$\begin{pmatrix} b & 0 & \ldots & 0 \\ 0 & \sigma(b) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma^{n-1}(b) \end{pmatrix}$$

   and let $z \in M_n(E)$ be the matrix

   $$\begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ \gamma & 0 & 0 & \ldots & 0 \end{pmatrix}$$

   where $0 \neq \gamma \in F$.

   Let A_γ be the subring of $M_n(E)$ generated by z and all $\bar{b}$ (for $b \in E$).

   (a) Show that $b \mapsto \bar{b}$ is an embedding, hence we can identify b with $\bar{b}$.

   (b) Show that $z^n = \gamma I$ and $zb = \sigma(b)z$ for all $b \in E$.

   (c) Show that any element a of A_γ has unique representation

   $$a = b_0 + b_1 z + \ldots b_{n-1} z^{n-1}$$

   with $b_i \in E$.

   (d) Prove that $A_\gamma \simeq D_\gamma$ are isomorphic as algebras over F.