Basic Notions in Algebra – Exercise No. 8

1. Let D be a division ring. The *characteristic* of D, denoted $\mathrm{char}(D)$ is defined as the minimal natural number $n \geq 1$ such that $n \cdot 1 = 1 + \cdots + 1$ (n times) is 0. If there is no such n, we set $\mathrm{char}(D) = 0$.

   (a) Show that $\mathrm{char}(D)$ is either 0 or a prime p.

   (b) Show that if $\mathrm{char}(D) = p > 0$, there is an embedding $\mathbb{F}_p \hookrightarrow D$.

2. Let D be a division ring with $\mathrm{char}(D) = p > 0$.

   Show that if $G \subset D^\times$ is a finite subgroup of the multiplicative group $D^\times$, then G is cyclic.

   Hint. Show that the set $E = \left\{\sum_{g\in G} a_g g \,:\, a_g \in \mathbb{F}_p \subset D\right\} \subset D$ is a finite division ring.

3. Let D be a division ring (of any characteristic).

   (a) Show that a finite commutative subgroup G of $D^\times$ is cyclic.
   Hint. Show that the division ring generated by G in D (that is, the intersection of all division rings in D containing G) is a field.

   (b) Give an example of a division ring D and a finite subgroup of $D^\times$ which is not cyclic.

4. Construction of a division ring D with $[D : Z(D)] = \infty$.

   Let K be a field and let $\sigma : K \to K$ be an automorphism. The ring of *twisted Laurent series*, $K((X,\sigma))$, is the set of Laurent series, i.e. elements of the form $\sum_{i=n}^{\infty} a_i X^i$ where $n \in \mathbb{Z}$ and $a_i \in K$, together with the following operations.

   Addition is defined as usual, that is

   $$\sum_i a_i X^i + \sum_i b_i X^i = \sum_i (a_i + b_i) X^i$$

   Multiplication is defined via the rule $Xb = \sigma(b)X$, that is,

   $$(\sum_i a_i X^i)(\sum_j b_j X^j) = \sum_{i,j} a_i \sigma^i(b_j) X^{i+j}$$

   (a) Show that the operations are well defined and that $D = K((X,\sigma))$ is a division ring.

   (b) Let $K_\sigma = \{a \in K : \sigma(a) = a\}$ be the fixed field of σ. Show that if σ is of infinite order ($\sigma^n \neq id$ for all n), then the center of D is K_σ, i.e. series of the form $a_0 X^0$ where $a_0 \in K_\sigma$.

   (c) Now take $K = \mathbb{Q}(t)$ to be the field of rational functions over $\mathbb{Q}$. Let $\sigma : K \to K$ be the automorphism defined by $t \mapsto 2t$, that is, σ maps a function $f(t) \in \mathbb{Q}(t)$ to $f(2t)$.
   Show that σ is an automorphism of K of infinite order, and that the fixed field K_σ of σ is $\mathbb{Q}$ (the constant functions).

(d) Deduce that if we set $D = \mathbb{Q}(t)((X, \sigma))$ with σ as in the previous paragraph, then D is a division ring, $Z(D) = \mathbb{Q}$ and $[D : Z(D)] = \infty$.

(e) Assume now that σ is of finite order n. Show that the center of $D = K((X, \sigma))$ is

$$Z(D) = \left\{ \sum_i a_i X^i \;:\; a_i \in K_\sigma \text{ and } a_i = 0 \text{ if } n \nmid i \right\}$$

Show also that $Z(D) \simeq K_\sigma((Y))$ (the field of Laurent series over K_σ) and $[D : Z(D)] = n^2$.

5. Chinese Remainder Theorem.

Let R be a commutative ring, and let I, J be two ideals in R.

(a) Show that the mapping $R/(I \cap J) \to R/I \times R/J$ induced by $r \mapsto (r + I, r + J)$ is injective.

(b) We say that I, J are *coprime* if $I + J = R$. Show that if I, J are coprime then the mapping $R/(I \cap J) \to R/I \times R/J$ is an isomorphism of rings.

(c) Let $I_1, I_2, \ldots, I_n$ be ideals in R such that for any $i \neq j$, $I_i + I_j = R$. Prove that there is an isomorphism of rings

$$R/(\bigcap_{j=1}^{n} I_j) \simeq \prod_{j=1}^{n} R/I_j$$

(d) Let $n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$ where p_i are distinct primes. Deduce that $\mathbb{Z}/n\mathbb{Z} \simeq \prod_{i=1}^{r} \mathbb{Z}/p_i^{e_i}\mathbb{Z}$.

(e) Give an example of a commutative ring R and an infinite collection of ideals $\{I_\alpha\}$ such that $I_\alpha + I_\beta = R$ for all $\alpha \neq \beta$ but $R/\cap_\alpha I_\alpha \to \prod_\alpha R/I_\alpha$ is not surjective.