Basic Notions in Algebra – Exercise No. 7

1. Let R be a ring. We define the *center* of R, denoted $Z(R)$, by

$$Z(R) = \{z \in R : zr = rz \quad \forall r \in R\}$$

   (a) Show that $Z(R)$ is a commutative sub-ring of R.
   (b) Show that if R is simple (does not have non-trivial two-sided ideals) then $Z(R)$ is a field.
   (c) If $R_1, \ldots, R_n$ are rings, then $Z(R_1 \times \cdots \times R_n) = Z(R_1) \times \cdots \times Z(R_n)$.
   (d) Show that $Z(M_n(R)) \simeq M_n(Z(R))$.
   (e) Compute $Z(\mathbb{H})$ ($\mathbb{H}$ are the Quaternions).

2. Let D be a division ring.

   (a) Show that $Z(D)$ is a field.
   (b) If $D_1 \subseteq D$ is a division ring, show that D is a (left) vector space over D_1, hence the dimension $[D : D_1] := \dim_{D_1} D$ is well-defined (may be ∞).
   (c) If $D_2 \subseteq D_1 \subseteq D$ are division rings and $[D : D_1]$, $[D_1 : D_2]$ are finite, show that $[D : D_2] = [D : D_1][D_1 : D_2]$.
   (d) If $[D : Z(D)] < \infty$ and $x \in D$, then the sub-ring in D generated by $Z(D)$ and x is a field.
   (e) If D is not a field then $[D : Z(D)] \geq 4$.
   (f) If $Z(D)$ is algebraically closed and $[D : Z(D)] < \infty$ then $D = Z(D)$.

3. Let p, q be positive rational numbers.

   Denote by $\mathbb{H}_{\mathbb{Q}}^{p,q}$ be the four dimensional $\mathbb{Q}$-vector space spanned by elements $1, i, j, k$ (that is, the space $\{a + bi + cj + dk : a, b, c, d \in \mathbb{Q}\}$).

   Denote by $\mathbb{H}_{\mathbb{R}}^{p,q}$ be the four dimensional $\mathbb{R}$-vector space spanned by $1, i, j, k$ (that is, the coefficients are in $\mathbb{R}$).

   (a) Show that the following multiplication rules

$$\begin{array}{lll} i^2 = -p & j^2 = -q & k^2 = -pq \\ ij = k & jk = qi & ki = pj \\ ji = -k & kj = -qi & ik = -pj \end{array}$$

   (where 1 acts as identity) equip $\mathbb{H}_{\mathbb{Q}}^{p,q}$ and $\mathbb{H}_{\mathbb{R}}^{p,q}$ with a structure of division algebras over $\mathbb{Q}$, $\mathbb{R}$ respectively.
   (b) Show that if $p', q' \in \mathbb{Q}$ are positive and $p' = a^2 p$, $q' = b^2 q$ for $a, b \in \mathbb{Q}$ then $\mathbb{H}_{\mathbb{Q}}^{p,q} \simeq \mathbb{H}_{\mathbb{Q}}^{p',q'}$.
   (c) Show that $\mathbb{H}_{\mathbb{R}}^{p,q} \simeq \mathbb{H}$ (the usual Quaternions).
   (d) Show that the algebra $\mathbb{H}_{\mathbb{Q}}^{1,2}$ contains two non-isomorphic maximal fields $\mathbb{Q}(i)$ and $\mathbb{Q}(j)$.
   (e) Show that for any $a, b, c \in \mathbb{Q}$, the field $\mathbb{Q}(\sqrt{-a^2p - b^2q - c^2pq})$ is contained in $\mathbb{H}_{\mathbb{Q}}^{p,q}$.