Basic Notions in Algebra – Exercise No. 6

1. Let R be a ring.

   (a) Let M be an R-module, and assume that there is a sequence of submodules M_i $(0 \le i \le n)$ such that
   $$(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$
   and each $M_{i+1} - M_i$ $(0 \le i < n)$ has the minimum condition.
   Show that M has the minimum condition.

   (b) Deduce that if $M = \bigoplus_{i=1}^{n} M_i$ and each M_i has the minimum condition then M has the minimum condition.

   (c) Let $\{M_i\}_{i=1}^{\infty}$ be an infinite sequence of nonzero R-modules. Show that $\bigoplus_{i=1}^{\infty} M_i$ does not have the minimum nor the maximum condition.

2. Let $\mu_n = \{z \in \mathbb{C} : z^n = 1\}$, and consider the subset
   $$\mu_{p^\infty} = \bigcup_{n=0}^{\infty} \mu_{p^n}$$

   (a) Show that μ_{p^∞} is an Abelian group.

   (b) Show that μ_{p^∞} does not have the maximum condition for subgroups.

   (c) Show that the only subgroups of μ_{p^∞} are μ_{p^n} $(n \ge 0)$ and μ_{p^∞}. Deduce that μ_{p^∞} has the minimum condition for subgroups.

   (d) Show that the homomorphisms $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ defined for $n \le m$ by $x \mapsto p^{m-n}x$ constitute a directed system and that
   $$\mu_{p^\infty} \simeq \varinjlim \mathbb{Z}/p^n\mathbb{Z}$$

3. Let μ_n be as in the previous question. Show the following claims:

   (a) $G = \bigcup_{n=1}^{\infty} \mu_n$ is an Abelian group.

   (b) $G \simeq \mathbb{Q}/\mathbb{Z}$.

   (c) $G \simeq \bigoplus_p \mu_{p^\infty}$ where p runs over all prime numbers.

   (d) G does not have the minimum nor the maximum condition for subgroups.

   (e) G is torsion (that is, for any $x \in G$ there is $n \in \mathbb{Z}$ such that $nx = 0$).

   (f) G is divisible (that is, for any $y \in G$ and $0 \ne n \in \mathbb{Z}$ there is $x \in G$ such that $y = nx$).

   (g) Give an example of a divisible (Abelian) group which is torsion free (that is, if $nx = 0$ for $x \ne 0$ then $n = 0$).

   (h) * Discuss the similarities between G and the torsion submodule of V, $\bigcup_{f \in F[x]} V_f$ constructed in Question 1 in Exercise 5.

4. Following my last class, here is an explanation of short exact sequences.

   A sequence of modules $\{M_i\}_{i=0}^n$ and mappings $f_i : M_i \to M_{i+1}$ is *exact* if for each $i > 0$, $\ker f_i = \operatorname{im} f_{i-1}$. We denote this by

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \ldots M_{n-1} \xrightarrow{f_{n-1}} M_n$$

   A *short exact sequence* of modules is an exact sequence

$$0 \xrightarrow{0} M' \xrightarrow{i} M \xrightarrow{p} M'' \xrightarrow{0} 0 \tag{1}$$

   (a) Show that a sequence in (1) is exact if and only if i is a monomorphism, p is an epimorphism and $\ker p = \operatorname{im} i$.

   (b) Let M be a module, N a submodule. Let $i : N \to M$ be the inclusion mapping $i(n) = n$ and let $p : M \to M - N$ be the natural projection $p(m) = m + N$. Show that the sequence

$$0 \to N \xrightarrow{i} M \xrightarrow{p} N - M \to 0$$

   is exact.

   (c) Conversely, if $0 \to M' \xrightarrow{j} M \xrightarrow{q} M'' \to 0$ is a short exact sequence, show that there exists a diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "j"] \arrow[d, "j"'] & M \arrow[r, "q"] \arrow[d, "id_M"'] & M'' \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & N \arrow[r, "i"] & M \arrow[r, "p"] & N - M \arrow[r] & 0
\end{tikzcd}$$

   where N is a submodule of M, i, p are as in the previous question, $r : M' \simeq M - N$ is an isomorphism, and all squares commute (that is, $j = ij$ and $rq = p$).

   (d) Let M_1, M_2 be two modules. Construct a short exact sequence

$$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$$

   (e) Give an example of four modules M', M'', M_1, M_2 such that there exist short exact sequences

$$0 \to M' \to M_1 \to M'' \to 0$$
$$0 \to M' \to M_2 \to M'' \to 0$$

   but $M_1 \not\simeq M_2$.