Basic Notions in Algebra – Exercise No. 5

1. Let F be a field, and let V be the vector space of (one-sided) infinite sequences over F,

$$V = \{(x_0, x_1, x_2, \dots) : x_i \in F\}$$

where addition and multiplication by scalar are done coordinate-wise.

Let $T : V \to V$ be the *shift* transformation

$$T((x_0, x_1, x_2, \dots)) = (x_1, x_2, x_3, \dots)$$

and let V^T be the $F[x]$-module as defined in Question 3, Homework 2; that is, $fv = f(T)v$ for any $f \in F[x], v \in V$.

For any $f \in F[x]$, define

$$V_f = \{v \in V : fv = 0\}$$

Show the following claims:

(a) V_f is a submodule of V and $\dim V_f = \deg f$.

(b) If $f \mid g$, then $V_f \subseteq V_g$.

(c) If f, g are polynomials, then $V_f \cap V_g = V_{\gcd(f,g)}$, $V_f + V_g = V_{\mathrm{lcm}(f,g)}$.

(d) If $f = g_1^{e_1} g_2^{e_2} \dots g_r^{e_r}$ where g_i are irreducible then $V_f = \bigoplus_i V_{g_i^{e_i}}$.

(e) V_f is simple if and only if f is irreducible.

(f) V_f is semi-simple if and only if f is a product of distinct irreducible polynomials.

(g) If $W \subseteq V$ is a finite dimensional subspace with $T(W) \subseteq W$ then there exists a polynomial f such that $W = V_f$.

(h) The mapping $(f) \mapsto V_f$ is one-to-one, onto, order-reversing, between the lattice of ideals of $F[x]$ and lattice of finite-dimensional submodules of V^T.

(i) $V_f \simeq F[x]/(f)$ as $F[x]$-modules.

(j) If $v, v' \in V$ are sequences, define their pointwise *product* vv' by $(vv')_i = v_i v'_i$ for any $i \geq 0$. If f, g are polynomials, let

$$W = V_f \cdot V_g = \{vv' : v \in V_f, v' \in V_g\}$$

Show that W is a finite dimensional T-invariant subspace, and deduce that there exists a polynomial h such that $W = V_h$.

(k) Let $\{F_n\}$ be the sequence defined (over $\mathbb{R}$) by $F_0 = 1$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2} + 2\sqrt{F_{n-1}F_{n-2}} \quad \text{for } n \geq 2$$

Show that the sequence $\{\sqrt{F_n}\}$ is the Fibonacci sequence and that $\{F_n\}$ satisfies a *recurrence relation*, i.e. there exist $k \geq 1$ and numbers $a_1, \dots, a_k$ such that for any $n \geq k$,

$$F_n = \sum_{i=1}^{k} a_i F_{n-i}$$

(l) V (as $F[x]$-module) does not have the minimum nor the maximum condition for submodules.

(m) Let $V_{x^\infty} = \bigcup_{n=0}^{\infty} V_{x^n}$. Then V_{x^∞} has the minimum condition but not the maximum condition for submodules.

2. A commutative ring is a *domain* if it has no zero divisors, that is, if $ab = 0$ then at least one of a, b is 0.

   (a) A finite domain is a field.

   (b) A domain does not contain any idempotents $\neq 0, 1$.

   (c) Use the isomorphism $R \simeq \mathrm{End}_R(R)$ and Question 3 in Homework 4 to show that if R is a semi-simple ring which is not a division ring then R contains idempotents $\neq 0, 1$.

   (d) Show that a domain which is not a field is not a semi-simple ring.

   (e) Show that a commutative ring is semi-simple if and only if it is a finite product of fields.

   Hint. Use the structure theorem.

3. (a) Show that $\mathbb{Z}$ is not a semi-simple ring.

   (b) What are the submodules of $\mathbb{Z}$? Are they simple, semi-simple?

   (c) Does $\mathbb{Z}$ have the maximum condition? the minimum condition?

   (d) Consider the quotients $C_n = \mathbb{Z}/n\mathbb{Z}$. Determine when they are simple, semi-simple $\mathbb{Z}$-modules. Also determine when they are simple, semi-simple rings.

   (e) Give an example of a direct product of simple modules which is not a semi-simple module (such a product must be infinite, why?).