Basic Notions in Algebra – Exercise No. 4

1. In this exercise we show that the notion of rank of a free module is not well-defined for non-commutative rings.

   Let F be a field, V be the infinite dimensional vector space over F with countable base $\{x_n\}_{n=0}^{\infty}$. Let $R = \mathrm{End}_F(V)$ be the ring of endomorphisms (which you computed in Homework No. 3). Denote the action of $r \in R$ on a vector v by vr. Let

$$A = \{r \in R : x_{2n}r = 0 \text{ for all } n \geq 0\}$$
$$B = \{r \in R : x_{2n+1}r = 0 \text{ for all } n \geq 0\}$$

   (a) Show that A, B are right ideals of R.
   (b) Show that $R = A \oplus B$ (that is, $R = A + B$ and $A \cap B = (0)$).
   (c) Show that $A \simeq R$, $B \simeq R$ as right R-modules.
   (d) Deduce that $R \simeq R \oplus R$.

2. (a) Let R be a ring, M a right R-module. Show that M is simple if and only if for any $0 \neq x \in M$, $y \in M$, there exists $r \in R$ such that $y = xr$.
   (b) Show that if F is a field and V is a vector space over F of any dimension, V is simple as $\mathrm{End}_F(V)$-module.

3. Let R be a ring, M a right R-module, $R' = \mathrm{End}_R(M)$.

   (a) An *idempotent* in R' is an element $e \in R'$ for which $e^2 = e$.
   Show that R' contains an idempotent $e \neq 0, 1$ if and only if M decomposes as $M = N \oplus N'$ for non-zero R-modules N, N'.
   Deduce that if R' does not contain idempotents $\neq 0, 1$ then M is *indecomposable* (cannot be written as direct sum of two non-trivial submodules).
   (b) Give an example of an indecomposable module M_R that is not simple. Compute also $\mathrm{End}_R(M)$.

4. Let R be a ring, $\{M_i\}_{i \in I}$ a collection of right R-modules, N a right R-module.

   (a) Show that $\mathrm{Hom}_R(\bigoplus_{i \in I} M_i, N) \simeq \prod_{i \in I} \mathrm{Hom}_R(M_i, N)$.
   (b) Show that $\mathrm{Hom}_R(N, \prod_{i \in I} M_i) \simeq \prod_{i \in I} \mathrm{Hom}_R(N, M_i)$
   (c) Deduce that if $\{N_j\}_{j \in J}$ is a collection of right R-modules then

$$\mathrm{Hom}_R(\bigoplus_{i \in I} M_i, \prod_{j \in J} N_j) \simeq \prod_{i,j} \mathrm{Hom}_R(M_i, N_j)$$

   (d) Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a finite sum. Assume that $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$. Then

$$\mathrm{End}_R(M) \simeq \prod_{i=1}^{n} \mathrm{End}_R(M_i)$$