Basic Notions in Algebra – Exercise No. 3

1. Let V be an infinite dimensional vector space over a field F.
   (a) Describe the endomorphism ring $R = \mathrm{End}_F(V)$ (as infinite matrices having certain properties).
   (b) Show that R is not a simple ring. (You need to construct a non-trivial ideal in R).
   Remark. In the previous homework we saw that $\mathrm{End}_F(V) \simeq M_n(F)$ is a simple ring for *finite* dimensional V.

2. Let R be a ring, $I \triangleleft R$ an ideal, M_R a right R-module. Define
$$MI := \{\sum_{i=1}^{n} m_i a_i : m_i \in M, a_i \in I\}$$
   (a) Show that MI is a sub-module of M, hence the difference (quotient) module $M - MI$ is well-defined.
   (b) Show that the set of cosets M/MI is an R/J-module for any ideal $J \subseteq I$.
   (c) For any $x \in M$ define the *annihilator of* x,
$$\mathrm{ann}_R\, x = \{a \in R : xa = 0\}$$
   Show that $\mathrm{ann}_R\, x$ is a right ideal of R.
   (d) Let $\mathrm{ann}_R\, M := \{a \in R : Ma = 0\}$ be the annihilator of M. Show that
$$\mathrm{ann}_R\, M = \bigcap_{x \in M} \mathrm{ann}_R\, x$$
   and that $\mathrm{ann}_R\, M$ is a two-sided ideal of R, which is the kernel of the ring homomorphism $\varphi : R^{op} \to \mathrm{End}_{\mathbb{Z}} M$ given by $\varphi(r)(m) = mr$.
   (e) Deduce that M is a $R/\mathrm{ann}_R\, M$ module.

3. Let M be a free R-module, that is $M \simeq R^I = \oplus_{i \in I} R$. The *rank* of M is defined as the cardinality of I.
   Show that if R is commutative, the rank is well defined, i.e. if $R^I \simeq R^J$ then $|I| = |J|$.
   Hint. Reduce to the case of vector spaces over a field.

4. Let R be a ring, M an R-module.
   (a) We say that M is *cyclic* if there exists $x \in M$ such that $M = xR$. Show that if M is cyclic then there exists a right ideal I of R such that $M \simeq R - I$ as right R-modules.
   (b) Assume M is simple. Then there exists a maximal right ideal I of R such that $M \simeq R - I$.
   (c) Conversely, for any maximal right ideal I of R, the module $R - I$ is simple.

(d) A ring R viewed as a right module R_R over itself is simple if and only if R is a division ring.

5. Let F be a field, and consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$$

Let $M = F^2$. Then R acts on M by

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} ax & bx + cy \end{pmatrix}$$

(a) Verify that R is a ring and that M with the above action is a right R-module.

(b) Describe all sub-modules of M_R.

(c) Is M_R simple? semi-simple?

(d) Compute $\mathrm{End}_R M$.