Basic Notions in Algebra – Exercise No. 2

1. **Ultraproducts**

   Let I be an index set and let $\mathcal{F}$ be a filter on I. Let $\{R_i\}_{i\in I}$ be a collection of rings (with 1). Define a relation $\sim$ on the product $R := \prod_i R_i$ by

   $$(a_i)_{i\in I} \sim (b_i)_{i\in I} \quad \text{if} \quad \{i : a_i = b_i\} \in \mathcal{F}$$

   (a) Show that this is an equivalence relation on R.

   (b) Show that if $a \sim a'$ and $b \sim b'$ then $a + b \sim a' + b'$ and $ab \sim a'b'$.

   Hence one can define a ring structure on the set of equivalence classes $R/\sim$. This ring is called the $\mathcal{F}$*-reduced product* of the R_i and denoted by $\prod_{i\in I} R_i/\mathcal{F}$. If $\mathcal{F}$ is an ultrafilter, the reduced product is called an *ultraproduct*.

   (c) Show that if $\mathcal{F}$ is an ultrafilter and each R_i is a division ring then the ultraproduct $\prod_i R_i/\mathcal{F}$ is also a division ring.

   (d) Show that if $\mathcal{F}$ is a principal filter with base S then $\prod_{i\in I} R_i/\mathcal{F} \simeq \prod_{i\in S} R_i$.

2. **Matrix Rings**

   Let R be a ring, and let $M_n(R)$ be the ring of $n \times n$ matrices with entries in R.

   (a) Let M be a (right) R-module. Let $M^{(n)}$ be the direct sum of n copies of M. Define a structure of a (right) $M_n(R)$-module on $M^{(n)}$.

   Hint. Consider the action of a matrix on a row vector.

   (b) Show that if $f : M \to N$ is an R-module homomorphism, there is an induced $M_n(R)$-module homomorphism $f^{(n)} : M^{(n)} \to N^{(n)}$. Show also that $(fg)^{(n)} = f^{(n)}g^{(n)}$.

   Remark. In the language of categories, the mappings $M \mapsto M^{(n)}$ and $f \mapsto f^{(n)}$ define a *functor* from $\mathfrak{Mod} - R$ to $\mathfrak{Mod} - M_n(R)$.

   (c) Let $m, n \geq 1$. Then $M_{mn}(R) \simeq M_m(M_n(R))$.

   (d) If $I \triangleleft R$ then the set $M_n(I)$ defined by

   $$M_n(I) := \{A = (a_{ij}) : a_{ij} \in I \text{ for all } i, j\}$$

   is an ideal of $M_n(R)$.

   (e) If $J \triangleleft M_n(R)$ then there exists $I \triangleleft R$ such that $J = M_n(I)$.

   Hence there is a bijective correspondence between ideals of R and ideals of $M_n(R)$ given by $I \mapsto M_n(I)$.

   (f) Deduce that if F is a field then the ring $M_n(F)$ is *simple*.

   (g) If $I \triangleleft R$ then $M_n(R)/M_n(I) \simeq M_n(R/I)$.

   (h) Show that $M_n(R[x]) \simeq M_n(R)[x]$.

3. **Polynomials**

Let V be a vector space over a field F. Let $T : V \to V$ be a linear transformation ($=$ F-module endomorphism of V). Define an action of the ring $F[x]$ on V as follows: if $v \in V$ and $\sum c_i x^i \in F[x]$, then

$$(\sum_i c_i x^i)(v) := \sum_i c_i T^i(v)$$

(a) Show that the above definition makes V into a left (and right) $F[x]$-module.

We denote by V^T the $F[x]$-module V.

(b) Give a condition on a subspace $W \subset V$ to be a sub-$F[x]$-module of V^T.

(c) Let $S : V \to V$ be another transformation and consider the $F[x]$-module V^S. Show that $V^S \simeq V^T$ (isomorphism of $F[x]$-modules) if and only if there exists an invertible linear transformation $\varphi : V \to V$ such that $S = \varphi T \varphi^{-1}$.