Basic Notions in Algebra – Exercise No. 13

1. Let V, V' be vector spaces over a field k. A *tensor product* of V, V' is a pair (W, τ) where W is a vector space over k and $\tau : V \times V' \to W$ is a bi-linear form such that the following property holds:

   If $\beta : V \times V' \to k$ is any bi-linear form, there exists a *unique* linear transformation $\varphi : W \to k$ such that $\varphi \circ \tau = \beta$ on $V \times V'$.

$$\begin{tikzcd} V \times V' \arrow[r, "\beta"] \arrow[d, "\tau"'] & k \\ W \arrow[ur, dashed, "\exists ! \varphi"'] & \end{tikzcd}$$

   (a) Prove that a tensor product (if it exists) is unique up to unique isomorphism

   So we can speak on *the* tensor product of V, V' and denote it by $V \otimes V'$. The element $\tau(v, v') \in V \otimes V'$ is denoted by $v \otimes v'$.

   (b) Let us now construct a tensor product of V, V' in the following way. Let U be a vector space over k with a basis $e_{v,v'}$ indexed over all pairs $(v, v') \in V \times V'$ (U is a huge space, you may think of it as functions $V \times V' \to k$ with finite "support"). Let U_0 be the subspace of U spanned by the following vectors:

$$\begin{array}{ll} e_{v_1+v_2,v'} - e_{v_1,v'} - e_{v_2,v'} & v_1, v_2 \in V, v' \in V' \\ e_{v,v'_1+v'_2} - e_{v,v'_1} - e_{v,v'_2} & v \in V, v'_1, v'_2 \in V' \\ e_{\lambda v,v'} - \lambda e_{v,v'} & v \in V, v' \in V', \lambda \in k \\ e_{v,\lambda v'} - \lambda e_{v,v'} & v \in V, v' \in V', \lambda \in k \end{array}$$

   Let $\overline{e_{v,v'}}$ be the image of $e_{v,v'}$ in the quotient U/U_0.

   Show that the quotient space $W = U/U_0$ together with the bi-linear (prove!) map $\tau : (v, v') \mapsto \overline{e_{v,v'}}$ is a tensor product of V, V'.

   (c) When the spaces V, V' are finite dimensional, it is easy to explicitly write a basis for $V \otimes V'$. Namely, let $e_1, \ldots, e_n$ be a basis of V, $e'_1, \ldots, e'_m$ a basis of V'. Show that $e_i \otimes e'_j$ $(1 \le i \le n, 1 \le j \le m)$ is a basis of $V \otimes V'$.

2. (a) Let $T : V \to W$, $T' : V' \to W'$ be linear transformations. Show that there is a unique linear transformation $S : V \otimes V' \to W \otimes W'$ satisfying $S(v \otimes v') = T(v) \otimes T(v')$ for all $v \in V, v' \in V'$.

   The transformation S is denoted by $T \otimes T'$.

   (b) Let $\{e_i\}, \{e'_j\}$ be bases of V, V'. Describe the matrix of $T \otimes T'$ with respect to the basis $\{e_i \otimes e'_j\}$ in terms of the matrices of T, T'.

   (c) What is the trace of $T \otimes T'$?

   (d) Let G be a finite group and let $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ be two representations. Show that there is a unique representation $\rho \otimes \rho' : G \to GL(V \otimes V')$ satisfying $(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$ for all $g \in G$.

(e) Show that $\chi_{\rho\otimes\rho'}(g) = \chi_\rho(g) \cdot \chi_{\rho'}(g)$ for all $g \in G$.

3. Let V be a finite dimensional vector space over a field k of characteristic zero. Let $\{e_i\}$ be a basis of V.

   (a) Show that there is a unique linear transformation $\theta : V \otimes V \to V \otimes V$ satisfying $\theta(v \otimes v') = v' \otimes v$ for all $v, v' \in V$.

   (b) Show that $V \otimes V$ decomposes as $S^2V \oplus \wedge^2 V$ where

$$S^2V = \{w \in V \otimes V : \theta(w) = w\} \qquad \text{(symmetric tensors)}$$
$$\wedge^2V = \{w \in V \otimes V : \theta(w) = -w\} \qquad \text{(anti-symmetric tensors)}$$

   (c) Compute $\dim S^2V$, $\dim \wedge^2 V$.

   (d) Let G be a finite group, $\rho : G \to GL(V)$ a representation. We know from previous question that there exists a representation $\rho \otimes \rho : G \to GL(V \otimes V)$.

   Show that $S^2V, \wedge^2 V$ are G-invariant subspaces, hence $\rho \otimes \rho$ restricts to representations $S^2(\rho)$ and $\wedge^2(\rho)$.

   (e) Compute the characters $\chi_{S^2(\rho)}, \chi_{\wedge^2(\rho)}$ in terms of the character χ_ρ.

4. Let G be a finite group acting on a finite set X. We already know that this action induces a representation $\rho = \rho_X$ of G on the space V of functions $f : X \to \mathbb{C}$ by $\rho(g)(f)(x) = f(g^{-1}x)$.

   Let σ be the trivial representation of G, and denote by χ_X the character of ρ_X.

   (a) Show that the multiplicity of σ in ρ is equal to the number of orbits of X.

   (b) Show that for any $g \in G$, $\chi_X(g)$ is equal to the number of fixed points of g, i.e. $\chi_X(g) = \#\{x \in X : gx = x\}$.

   (c) Let G act on two finite sets X_1, X_2. Then G acts on $X_1 \times X_2$ by $g(x_1, x_2) = (gx_1, gx_2)$. Show that the corresponding representations satisfy $\rho_{X_1\times X_2} \simeq \rho_{X_1} \otimes \rho_{X_2}$.

   (d) In particular, G acts on $X \times X$ by $g(x, y) = (gx, gy)$. Show that $\chi_{X\times X} = \chi_X^2$.

   (e) Let $|X| > 1$. We say that G acts *doubly transitively* if for all $x, y, x', y' \in X$ with $x \neq y$, $x' \neq y'$, there exists $g \in G$ with $gx = x'$ and $gy = y'$. Show that the following conditions are equivalent:

      i. G acts doubly transitively on X.

      ii. The action of G on $X \times X$ has two orbits, one is the diagonal and the other is its complement.

      iii. $\rho_X \simeq \sigma \oplus \rho'$ where ρ' is an irreducible non-trivial representation of G.