Basic Notions in Algebra – Exercise No. 12

1. (a) Let G be an abelian group of order n and let k be an algebraically closed field of characteristic not dividing n. Show that the group ring $k[G]$ is isomorphic to the product $k \times k \times \cdots \times k$ (n times).

   (b) Show that $\mathbb{C}[\mathbb{Z}/4\mathbb{Z}] \simeq \mathbb{C}[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ are isomorphic group rings of non-isomorphic groups.

2. Let S_3 be the group of permutations on 3 elements.

   (a) Describe the group ring $\mathbb{C}[S_3]$ as a product of matrix rings.

   (b) Find all irreducible representations of S_3 (over $\mathbb{C}$) and write the character table.

   (c) What is the geometric meaning of the 2-dimensional irreducible representation of S_3 ?

3. Let G be a cyclic group of order n. Describe the irreducible representations and character table of G (over $\mathbb{C}$).

4. Let $\rho : G \to Aut(V)$ be a (finite dimensional) representation of G. Let $V^\vee$ be the dual space of V. Define the *contragredient* representation $\rho^\vee : G \to Aut(V^\vee)$ by $\rho^\vee(g)(\varphi)(v) = \varphi(\rho(g^{-1})(v))$ for $v \in V, \varphi \in V^\vee, g \in G$.

   Show that $\rho^\vee$ is a representation, and that if χ_ρ is the character of ρ, then $\chi_{\rho^\vee}(g) = \chi_\rho(g^{-1})$ for all $g \in G$.

5. Let R be a ring and let U be its group of units. Prove that if U is finite of odd order, then U is abelian and $|U| = \prod_{i=1}^{t}(2^{m_i} - 1)$ for some positive integers m_i and t.

   Hints:

   (a) Show that there is a ring homomorphism $\varphi : \mathbb{F}_2[U] \to R$.

   (b) Show that $\mathbb{F}_2[U]$ is semi-simple.

   (c) Show that the image of φ in R is a subring R' of R which is semi-simple and its group of units is equal to U.

   (d) Show that $R' \simeq \prod_{i=1}^{t} M_{n_i}(\Delta_i)$ where Δ_i are finite division rings, $n_i \geq 1, t \geq 1$.

   (e) Show that for all i, $|\Delta_i| = 2^{m_i}$ for some m_i.

   (f) Show that $n_i = 1$ for all i.

   (g) Complete the proof.

   (h) Deduce that there is no ring having exactly 5 units.