Basic Notions in Algebra – Exercise No. 11

1. **Actions arising from groups**

   Let G be a group.

   (a) Verify that the following three rules give actions of G on itself (as a set): $g \cdot x = gx$, $g \cdot x = xg^{-1}$, $g \cdot x = gxg^{-1}$.

   (b) Now let H be a subgroup. Verify that $g \cdot xH = (gx)H$ is an action on left cosets of H.

   (c) Verify that $g \cdot H = gHg^{-1}$ gives an action of G on its subgroups.

   (d) Describe the orbits, stabilizers and fixed points of all the above actions.

2. **Applications to group theory**

   (a) Let G be a group and let H be a subgroup of finite index in G ($[G : H] < \infty$). Show that there exists a *normal* subgroup $N \lhd G$ such that $N \subseteq H$ and $[G : N] < \infty$.
   Hint. Consider the action of G on left cosets of H.

   (b) Let G be a finite group of order p^n. Show that $Z(G) \neq \{1\}$.

   (c) Let G be a finite group of order divisible by p. Let

   $$X = \{(x_0, x_1, \ldots, x_{p-1}) \in G^p \, : \, x_0 x_1 \cdot \cdots \cdot x_{p-1} = 1\}$$

   By considering the action of $\mathbb{Z}/p\mathbb{Z}$ on X via "rotations"

   $$1 \cdot (x_0, x_1, \ldots, x_{p-1}) = (x_1, x_2, \ldots, x_{p-1}, x_0)$$

   (verify this!) show that G contains an element of order p.

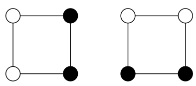
3. **Regular representation**

   (a) Let G act on a set X, and let k be a field. Show that G acts on the set of functions $f : X \to k$ by $(gf)(x) = f(g^{-1}x)$.

   (b) Let G act on itself by left multiplication. Show that in the induced action of (a) on the group algebra $k[G]$, the action of g is identical to (left) multiplication by the element $1 \cdot g \in k[G]$. This representation of G is called the *regular representation*.

4. **Some combinatorics**

   A regular polygon has n vertices. Each vertex can be colored either white or black. A coloring of the polygon is a coloring of (all) its vertices. We identify two colorings if one can be obtained from the other by rotating the polygon. The following picture shows equivalent colorings for $n = 4$:

   Show that the number of colorings is $\frac{1}{n} \sum_{d|n} \varphi(n/d) 2^d$ where $\varphi(m)$ is the *Euler function* counting the number of nonnegative integers not larger than m which are relatively prime to m.