Basic Notions in Algebra – Exercise No. 10

Let k be a field, G a group.

1. The *group algebra* $k[G]$ is defined as the set of functions $f : G \to k$ such that for almost all $x \in G$, $f(x) = 0$ (i.e. the set $\{x : f(x) \neq 0\}$ is finite), with the following operations.

   Addition is done pointwise, $(f + g)(x) := f(x) + g(x)$; multiplication is defined as the *convolution*

   $$(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x)$$

   Show that these operations are well-defined, and that $k[G]$ is an associative algebra over k.

   Show also that k is embedded in $k[G]$ via the mapping $a \mapsto f_a$ where $f_a : G \to k$ is defined by $f_a(1) = a$, $f_a(x) = 0$ for $x \neq 1$.

2. Show that $k[G]$ can also be viewed as a vector space over k with basis elements indexed by G, call them $\{x\}$, so that $k[G]$ is the set of expressions $\sum_{x \in G} a_x \cdot x$ for $a_x \in k$, with multiplication defined by the rule

   $$(a \cdot x)(b \cdot y) = ab \cdot (xy)$$

3. (a) Let G be a cyclic group of order n. Show that $k[G]$ is isomorphic to the ring $k[x]/(x^n - 1)$.

   (b) Let G be infinite cyclic. Show that $k[G]$ is isomorphic to the ring of *Laurent polynomials*, i.e. rational functions in $k(x)$ of the form $f(x)x^n$ with $n \in \mathbb{Z}$, $f(x) \in k[x]$.

4. Let G, G' be groups and let $\psi : G \to G'$ be a group homomorphism.

   (a) Show that there is a unique k-algebra homomorphism $\tilde{\psi} : k[G] \to k[G']$ satisfying $\tilde{\psi}(1 \cdot x) = 1 \cdot \psi(x)$ for all $x \in G$.

   (b) Show that $\tilde{\psi}$ is one-to-one if and only if ψ is.

   (c) Show that $\tilde{\psi}$ is onto if and only if ψ is.

   (d) Describe $\ker \tilde{\psi}$.

5. Consider the trivial homomorphism $\varepsilon : G \to 1$. By the previous exercise it induces a k-algebra homomorphism $\tilde{\varepsilon} : k[G] \to k[1] \simeq k$. Its kernel is called the *augmentation ideal* of $k[G]$. Compute it.

6. Let k be of characteristic p and let p divide $|G|$ (the order of G). Show that the augmentation ideal of $k[G]$ has no (direct) complement (as a $k[G]$-module). Deduce that $k[G]$ is not semi-simple in this case.

7. Let k, G be arbitrary. Show that the center of $k[G]$ is the set of *class functions* on G; i.e. functions $f : G \to k$ such that $f(x) = f(y)$ if x, y are conjugate in G.