Basic Notions in Algebra – Exercise No. 1

Let $\mathcal{C}$ be a category, I a set, and let $A_i \in \mathcal{C}$ for all $i \in I$.

A *product* of $\{A_i\}_{i\in I}$ is an object A of $\mathcal{C}$ together with morphisms $p_i : A \to A_i$ for all i, such that if B is any object with morphisms $q_i : B \to A_i$ for all i then there exists a unique morphism $q : B \to A$ such that $q_i = p_i q$ for all i.

A *coproduct* of $\{A_i\}_{i\in I}$ is the product of the A_i in the opposite category $\mathcal{C}^{op}$. Namely, an object A with morphisms $u_i : A_i \to A$ such that if B is any object with morphism $v_i : A_i \to B$ then there exists a unique morphism $v : A \to B$ such that $v_i = vu_i$ for all i.

1. Construct products and coproducts in the category $R - \mathfrak{Mod}$.

   Hint. Consider the sets

   $$\prod_{i\in I} A_i := \{a = (a_i)_{i\in I} : a_i \in A_i\}$$
   $$\bigoplus_{i\in I} A_i := \{a = (a_i)_{i\in I} : \{i : a_i \neq 0\} \text{ is finite}\}$$

2. Construct products and coproducts in the categories $\mathfrak{Set}$, $\mathfrak{Grp}$.

   Let I be a partially ordered set and let $\{A_i\}$ be objects of $\mathcal{C}$. For each $i \leq j$ let $\varphi_{ij} : A_i \to A_j$ be morphisms such that $\varphi_{ii} = 1_{A_i}$ and $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$ for all $i \leq j \leq k$.

   A *direct limit* of the system $\{A_i, \varphi_{ij}\}$ is an object A and morphisms $\varphi_i : A_i \to A$ such that for all $i \leq j$ we have $\varphi_i = \varphi_j \varphi_{ij}$ and if B is any object with morphisms $\psi_i : A_i \to B$ satisfying $\psi_i = \psi_j \varphi_{ij}$ for every $i \leq j$ then there exists unique morphism $\psi : A \to B$ such that $\psi_i = \psi\varphi_i$ for all $i \in I$. We denote the direct limit by $\varinjlim A_i$.

3. A partially ordered set I is *directed* if for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Construct the direct limit over directed sets, in $\mathfrak{Grp}$, as follows (the same proof works for $R - \mathfrak{Mod}$).

   (a) Let $\coprod_{i\in I} A_i$ be the disjoint union of the A_i. Define a relation $a_i \sim a_j$ ($a_i \in A_i$, $a_j \in A_j$) if there exists $k \geq i, k \geq j$ such that $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$ (equality in A_k). Show that this is an equivalence relation.

   (b) Let $A = \coprod_i A_i / \sim$ be the set of equivalence classes. Define multiplication on classes as follows. If $[a_i]$ and $[a_j]$ are classes, $a_i \in A_i, a_j \in A_j$, let $k \geq i, j$ and set $[a_i] \cdot [a_j] = [\varphi_{ik}(a_i) \cdot \varphi_{jk}(a_k)]$.

   Show that this is well defined (does not depend on k nor on the representatives of the classes), and equips A with a structure of a group.

   (c) Define $\varphi_i : A_i \to A$ by $\varphi_i(a) = [a]$ for $a \in A_i$. Show that φ_i are group homomorphisms and $(A, \{\varphi_i\})$ is a direct limit of $\{A_i, \varphi_{ij}\}$.

   Again, let I be partially ordered. Let $\{A_i\}$ be a collection of objects with morphisms $\varphi_{ij} : A_j \to A_i$ for every $i \leq j$, such that $\varphi_{ii} = 1_{A_i}$ for all i and $\varphi_{ik} = \varphi_{ij}\varphi_{jk}$ for all $i \leq j \leq k$.

An *inverse limit* of $\{A_i, \varphi_{ij}\}$ is defined as $(A, \{\varphi_i\})$ where A is an object and $\varphi_i : A \to A_i$ morphisms such that $\varphi_i = \varphi_{ij}\varphi_j$ for every $i \leq j$, and for any object B with morphisms $\psi_i : B \to A_i$ satisfying $\psi_i = \varphi_{ij}\psi_j$ for every $i \leq j$, there exists a unique morphism $\psi : B \to A$ such that $\psi_i = \varphi_i \psi$ for every $i \in I$. The inverse limit is denoted by $\varprojlim A_i$.

4. Construct inverse limits in the category $R - \mathfrak{Mod}$.

   Hint. Consider appropriate subset of the product $\prod_{i \in I} A_i$.

5. Let G be a group, and let $\{G_i\}_{i \in I}$ be a collection of subgroups.

   (a) Define an order on I by $i \leq j$ if $G_i \subseteq G_j$. If $i \leq j$, let $\varphi_{ij} : G_i \to G_j$ be the inclusion map. Compute $\varinjlim G_i$.

   (b) Now define $i \leq j$ if $G_i \supseteq G_j$. If $i < j$, let $\psi_{ij} : G_j \to G_i$ be the inclusion map. Compute $\varprojlim G_i$.

   (c) Every group is a direct limit of finitely generated groups.

   (d) Now assume all G_i are normal in G and order I as in (a). Show that if $i \leq j$ then $\alpha_{ij} : G/G_j \to G/G_i$ are well defined, hence give rise to the inverse limit $G_I := \varprojlim G/G_i$. Show the existence of a mapping $\alpha : G \to G_I$ and give a necessary and sufficient condition (on the G_i) for α to be a monomorphism.

6. An abelian group G is called *torsion group* if for every $x \in G$ there exists $n \in \mathbb{Z}$ such that $nx = 0$.

   (a) Show that a direct sum of torsion groups is torsion.

   (b) Is the direct product of torsion groups also torsion? What about direct limits? inverse limits? Prove or give counter examples.

7. Let X be a set, $\mathcal{F}$ a filter on X. $\mathcal{F}$ is called *principal* if there exists $S \subset X$ such that $\mathcal{F} = \{A : A \supseteq S\}$. In this case S is called the *base* of $\mathcal{F}$. If $|S| = 1$ then $\mathcal{F}$ is called a *singleton*.

   Show the following:

   (a) A filter $\mathcal{F}$ is an ultrafilter iff for any $A \in X$ either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$.

   (b) A principal filter $\mathcal{F}$ is an ultrafilter iff it is a singleton.

   (c) If X is finite then any filter $\mathcal{F}$ on X is principal. Deduce that any ultrafilter on a finite set is a singleton.

   (d) If X is infinite then an ultrafilter $\mathcal{F}$ is not a singleton iff it contains the filter consisting of all the complements of finite subsets of X.